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Every recursive boolean algebra is isomorphic to one with incomplete atoms

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Abstract

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The theorem of the title is proven, solving an old question of Remmel. The method of proof uses an algebraic technique of Remmel–Vaught combined with a complex tree of strategies argument where the true path is needed to figure out the final isomorphism.

1. Introduction

In this paper, we solve an old question of Remmel [10, 11] by proving that for any recursive boolean algebra B_0 there is a recursive boolean algebra B_1 isomorphic to B_0 such that the atoms of B_1 are Turing incomplete. This result should be seen in the context of theoretic studies looking at the behavior under isomorphism of distinguished relations in recursive models, going back to, for instance, Ash and Nerode [1]. Remmel [10, 11] had earlier proven that if a boolean algebra B_0 has a recursive presentation with an infinite recursive set of atoms, then for any given r.e. degree, B_0 had a recursive copy whose atoms had degree **d**.

Modifying the difficult coding argument of Feiner [5, 6], Remmel, however, also showed that there exists a recursive boolean algebra B_0 whose atoms are *intrinsically nonlow*. That is, if B_1 is a recursive boolean algebra isomorphic to

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B_0 then the atoms of B_1 form a nonlow Π_1^0 set, and in particular are not recursive. Every recursive boolean algebra is the interval algebra of a recursive linear ordering. If $B = \text{Intal}(L)$, then the atoms of B correspond to the successivities of L. It is not difficult to show that if L is a linear ordering with an infinite recursive set of successivities, then there is a recursive L' isomorphic to L such that the successivities of L' have degree $\mathbf{d}$. Using this sort of reasoning many theorems concerning boolean algebra can be deduced by manipulating results on linear settings.

Our result cannot be so deduced. In [4] Downey and Moses constructed a recursive linear ordering L whose successivities were intrinsically complete; that is, all recursive linear orderings isomorphic to L have complete successivities.

Our result is proven by a tree of strategies priority argument. A crucial ingredient is Remmel's extension (from [10, 11]) of Vaught's theorem. We discuss such preliminaries in Section 2 and prove the main result in Section 3. We remark that the combination of an isomorphism construction with the Remmel–Vaught lemma has other applications. In particular, Carl Jockusch and the author [3] used this technique to prove that all low boolean algebras are isomorphic to recursive ones, thereby solving a question dating back to Feiner's 1967 thesis [5].

Notation is standard and follows Soare [12], Downey [2] and Monk [9].

2. Preliminaries

Let B be a recursive boolean algebra. It is well known that B is recursively isomorphic to a recursive subalgebra of $\bar{\mathbb{Q}}$, the atomless boolean algebra. This is shown in, for instance, Remmel [10, Theorem 1.2]. The same proof also shows that there is a recursive linear ordering L so that B is recursively isomorphic to $\text{Intal}(L)$, the interval subalgebra of left closed right open intervals of L. We will suppose that our boolean algebras are so presented as subalgebras of $\bar{\mathbb{Q}}$ given an interval algebra of recursive suborderings of the rationals. Furthermore, for convenience, we shall suppose that the orderings have endpoints.

We will need the following result of Remmel that extends one of Vaught.

2.1. Theorem (Remmel–Vaught [10, Theorem 2.1]). *Let A ve a subalgebra of $\bar{\mathbb{Q}}$ and suppose the atoms of B are infinite in number. Let $Atom(B) = \{d_0, d_1, \ldots\}$. For each i, suppose $e_1^i, \ldots, e_{k_i}^i$ are pairwise disjoint elements of $\bar{\mathbb{Q}}$ with $d_i = \bigwedge_{j=1}^{k_i} e_j^i$. Let C be the subalgebra of $\mathbb{Q}$ generated by B together with all the e_j^i. Then C is isomorphic to B.*

For our purposes, we work with orderings. Let $S(L)$ denote the collection of *successivities* (or adjacencies) of L.

2.2. Corollary. *Let L_1 and L_2 be suborderings of $\mathbb{Q}$ with infinitely many successivities. Suppose that there is an injective mapping $g: L_1 \to L_2$ which is order preserving and has the property that if $y \notin \operatorname{ra} g$ then there exist c, d in $\operatorname{ra} g$ such that $\|[c, d]\| < \infty$ and $[g^{-1}(c), g^{-1}(d)]$ is a successivity of L_1.*

Then $\operatorname{Intal}(L_1)$ *is isomorphic to* $\operatorname{Intal}(L_2)$.

3. The proof

In view of Lemma 2.2, to prove our main result, it will suffice to consider a recursive subordering $A = \{a_i : i \in \omega\}$ of $\mathbb{Q}$ and to construct a recursive linear subordering B of Q together with an isotone injection $f: A \to B$ with the property of Lemma 2.2 which we restate here for convenience.

3.1. *If $b \notin \operatorname{ra} f$ then there exist $c, d \in \operatorname{ra} f$ such that $c < b < d$, $\|[c, d]\| < \infty$ and $[f^{-1}(c), f^{-1}(d)]$ is a successivity in A.*

Let $S(L)$ denote the collection of pairs that constitute successivity of L. We build an r.e. set D and meet the requirements

$$R_{2e}: \quad e^{S(B)} \neq D.$$

Here we use e to represent the eth Turing procedure, and will similarly in the construction let at stage s, e^{D_s} denote $e_s^{D_s}$. We shall use a Fredberg type strategy to meet R_e: We pick a follower x, wait till

$$l(e, s) = \max\{x : (\forall y < x)(e^{S(B_s)}(x) = D_s(x) = 0)\}$$

and then act to diagonalize and preserve $S(B_s) = S(B)$. There may be infinitely many attacks on this requirement, through infinitely many x, but if this is the case then the requirement will meet by divergence. (More on this later.)

The most difficult part of the construction will be controlling the definition of f. In fact f will be defined in a Δ_3^0 way via the 'true path' of the construction. While this phenomenon is not unique (see e.g. Downey [2]), it is quite unusual. So we shall in fact construct a tree of partial injections with $f(a_i) = \lim_s f_{\delta,s}(a_i)$ where δ denotes the initial segment of the true path (TP) of the construction devoted to a_i. Hence to our list of requirements we add the additional ones:

$$R_{2e+1}: \quad \text{If } e(\delta) = 2e + 1 \text{ and } \delta \subseteq TP \text{ then } \lim_s f_{\delta,s}(a_e) = f_\delta(a_e) \text{ exists.}$$

Furthermore we must ensure that the function $f(a_i) = f_\delta(a_i)$ for $e(\delta) = 2e + 1$ and $\delta \subseteq TP$ satisfies 3.1.

To discuss the method whereby we achieve the goals above, we consider the situation for satisfying an R_{2e} in isolation, but in the 'α-correct' environment within which it will be working. Now R_{2e}, from some point onwards, will be living in an environment where there will be a finite number of points $c_1 < \cdots < c_n$

(including $a_1, \ldots, a_e$) of A such that, as far as R_{2e} is concerned, $f(c_i)$ is fixed. That is if δ is the correct node devoted to solving R_{2e}, then at δ-stages s (i.e., when δ looks correct) it will be the case that B_s must respect exactly the commitments:

(i) $f_{\delta,s}(c_i) = b_{j_i}$ (b_{j_i} fixed), and

(ii) (This will actually be implicit from (i), as we see.) If $[q, r]$ is declared to be a *preserved block* by some γ devoted to some R_{2k} of higher priority than δ, then δ is committed to keeping $[q, r]$ to be the same size in B_{s+1} as it is in B_s.

Diagram 1 below might be helpful for visualizing (i) and (ii) above; in a typical situation at a δ-stage s.

In Diagram 1, the lined arrows must be respected by R_{2e}, but the dotted ones can be shifted. Note also that the pullback of $[q, r]$ in B is a successivity in A_s. The idea here is that $[q, r]$ would be, in a manner we will see, devoted to meeting R_{2k} for some k.

As we noted earlier, the basic idea used to meet the R_{2e} is to use a Friedberg procedure. So we would like to pick an x, wait till $e^{S(B_s)}(x)\downarrow = 0$, put x into D and ensure that $S(B_s) = S(B)$. Unfortunately, ensuring that $S(B_s) = S(B)$ is a very difficult task, when combined with the isomorphism property. For instance, if we need to ensure for the sake of higher priority isomorphism conditions that $c_1 \to b_{j_1}$ and $c_2 \to b_{j_2}$ then if $[c_1, c_2]$ is, say dense, then $[b_{j_1}, b_{j_2}]$ in B must be dense too. Now if at a *finite* step s we try to preserve some successivity in $[b_{i_j}, b_{j_2}]_s$ we will fail. We need to guess at the behavior of A relative to the fixed points.

To make the description simpler, we now concentrate upon only one interval, namely $[c_1, c_2]$. Essentially we will try to guess the behavior and existence of successivity in A between c_1 and c_2. At the least refined level, we will need an outcome guessing that, at some s, $|[c_1, c_2]| = |[c_1, c_2]$ in $A_s|$. We write $[c_1, c_2]_s$ for '$[c_1, c_2]$ in A_s'. This outcome is guessing that $[c_1, c_2]$ is finite in A.

Now if this finite outcome f is truly the correct one then $S(B)$ contains all

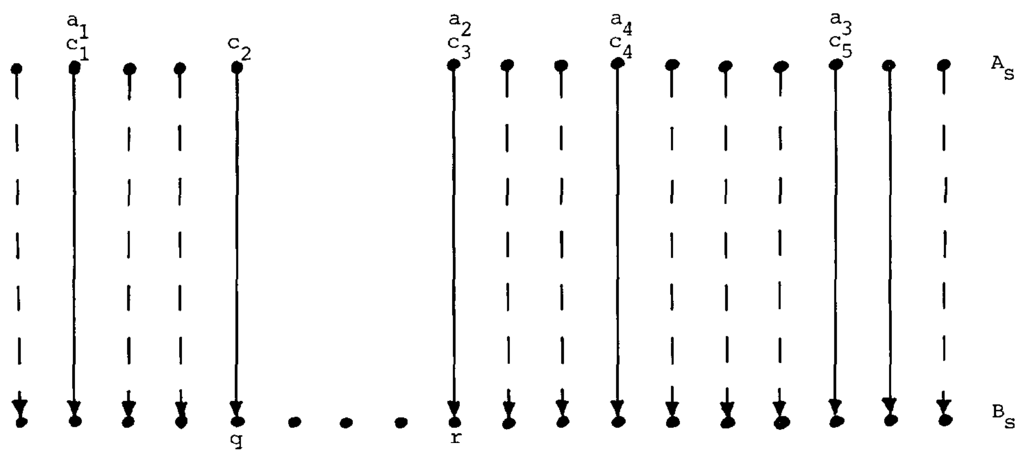

Diagram 1.

consecutive pairs in $S([c_1, c_2])$. If $e^{S(B)}(x) = D(x)$, then in this case we must see at some stage s a computation $e^{S(B_s)}(x) = D_s(x)$ where $S(B_s) = S(B)$ on the interval $[c_1, c_2] = [c_1, c_2]_s$. Hence a version of R_{2e} guessing this outcome f will believe that $[c_1, c_2]_s = [c_1, c_2]$. As a consequence, on $f([c_1, c_2])$, R_{2e} guessing f believe that $S(B_s) = S(B)$ (essentially).

All other versions of R_{2e} equipped with other guesses will wait for certain successivity to be killed in $[c_1, c_2]$ before they will be prepared to believe that $S(B_s) = S(B)$.

The remaining cases depend on whether $[c_1, c_2]$ contains a successivity, and whether neither, either or both $[c_1, x_1]$ or $[x_2, c_2]$ is a successivity for some x_1, x_2 in A. The reason that this is important can be seen considering the case of a version of R_{2e} guessing that $[c_1, c_2]$ is dense (outcome d). In this case this version of R_{2e} will never believe a computation which relies on $(x, y) \in S(B_s)$ with $f(c_1) \leq x < y \leq f(c_2)$. The reader should note that '$[c_1, c_2]$ dense' is Π_2 behavior. We can play the outcome d when this 'appears correct'. In this case one way to do this is to use a *test set* which we call test(c_1, c_2, d, s). At some stage s, we put all the current elements of $[c_1, c_2]_s$ into test(c_1, c_2, d, s). We keep test$(c_1, c_2, d, t) =$ test(c_1, c_2, d, s) until a stage $s_1 > s$ occurs where for all elements $x < y$ in test(c_1, c_2, d, s_1) $(=$ test$(c_1, c_2, d, s))$ there is an element $z = z(x, y)$ with $z \in [c_1, c_2]_{s_1}$ and $x < z < y$. At stage s_1 we could then play outcome d for $[c_1, c_2]$ and reset the test set as test$(c_1, c_2, d, s_1) = [c_1, c_2]_{s_1}$. Clearly if we play outcome d infinitely often, then $[c_1, c_2]$ is dense. Note that if $[c_1, c_2]$ is dense, at the end there will be an *isomorphism* between $[c_1, c_2]$ and $\{z \in B \mid f(c_1) \leq z \leq f(c_2)\}$.

For the remaining outcomes which are stronger than f, but weaker than d, the strategy is more intricate, and depends not only on the existence of successivities in $[c_1, c_2]$ but their overall location. Note that, as above, it is Π_2 to determine if c_1 and/or c_2 are limit points from respectively the right and/or the left. Namely for c_1, for instance, we will have a test set limit(c_1, s). This contains at a stage s the successor of c_1 at stage s. We issue another chip to the outcomes that believe c_1 is a limit point in $[c_1, c_2]$ if limit$(c_1, s) \neq$ limit$(c_1, s+1)$.

We will arrange the outcome in the following order of ascending priority.

f – the finite outcome that $[c_1, c_2]$ is finite,
(c_1, c_2) – that c_1 and c_2 are not limit points,
(c_1, ∞) – c_2 is a limit point but c_1 is not,
(∞, c_2) – c_1 is a limit point but c_2 is not,
(∞, ∞) – both c_1 and c_2 are limit points but $[c_1, c_2]$ contains a successivity,
d – $[c_1, c_2]$ is dense.

We can consider the outcomes between f and d as suboutcomes of s the outcome that believes there is a successivity in $[c_1, c_2]$.

The (c_1, c_2)*-strategy*. The (c_1, c_2)-strategy comes equipped with a current guess $(x_{1,s}, x_{2,s})$ as to the successors of c_1 and c_2 respectively. We play this outcome

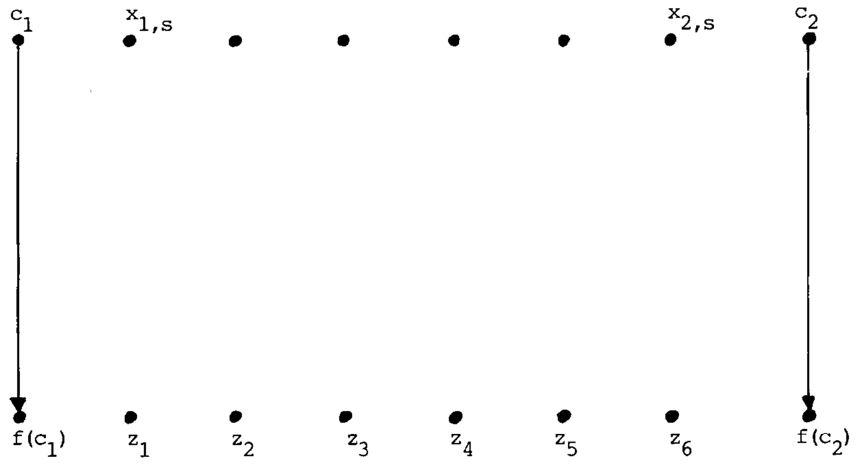

Diagram 2.

when d does not look correct, but a new element z enters $[c_1, c_2]_{s+1} - [c_1, c_2]_s$ and z is not in $[c_1, x_{1,s}) \cup (x_{2,s}, c_2]$. For this situation, an $e^{S(B_s)}(x)$ computation will only be believable if, upon $[c_1, c_2]_s$ the members of $S(B_s)$ are partitioned into two disjoint sets with no common endpoints, and including $f(c_1)$ and $f(c_2)$. A typical situation is given in Diagram 2 above.

The consistency condition for $S(B_s)$ is that for all pairs (p, q) of the form $(f(c_1), z_i)$, (z_i, z_j) $(j > i)$ or $(z_i, f(c_2))$ or $(f(c_1), f(c_2))$ with Gödel numbers below $u = u(e^\sigma; x) = 0$, we can believe $\sigma = S(B_s)$ iff

(i) $\#(p, q) \in \sigma$ iff $(p, q) \in S(B_s)$ for $\#(p, q) \leq u$, and

(ii) for some i, $\#(z_i, z_{i+1}) > u$.

The reason for (ii) is that we are, after all believing that $\|[c_1, c_2]\| = \infty$ and hence if the outcome is (c_1, c_2) then there will need to be infinitely many 'splitting' elements entering between $f(c_1)$ and $f(c_2)$.

The idea is to now force $S(B)$ to extend σ (should this be the correct outcome). To do this we find the least i with $\#(z_i, z_{i+1}) > u$ and map $x_{1,s}$ to z_i and $x_{2,s}$ to z_{i+1}. Adding $\|[c_1, c_2]\| - 4$ new elements (with large Gödel numbers) between z_3

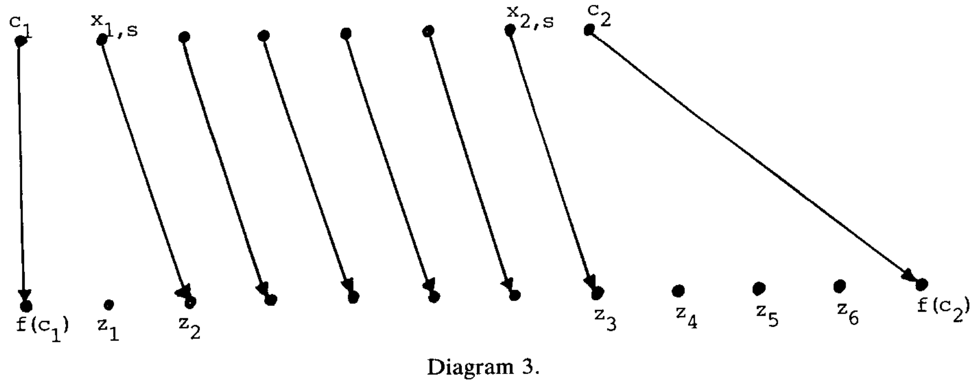

Diagram 3.

and z_4 to serve as images for the remaining elements of $[c_1, c_2]$. (See Diagram 3 in the case $c = 2$.)

The reader should note that the elements z_i are no longer in the range of f_s. If it is indeed the case that $x_1 = x_{1,s}$ and $x_2 = x_{2,s}$ then the fact that the z_i are no longer in the range of f is fine since, in essence, all we are doing is splitting the successivities $[c_1, x_1]$ and $[x_2, c_2]$ finitely. Thus we get a temporary win by restraining the definition of f_s as above (and so keeping $S(B_s)$ extending σ) and putting x into D, causing a disagreement. The strategy above can only be injured if at least one of the stage s successivities ($[c_1, x_{1,s}]$ and $[x_{2,s}, c_2]$) contain a nonsuccessivity in B. So we cancel the maps above if there occurs a number a entering $[c_1, c_2]_t - [c_1, c_2]_s$ with a splitting one of $[c_1, x_{1,s}]$ or $[x_{2,s}, c_2]$. After this entry of a occurs, to make the cominatorics simpler, we agree that we won't play the $[c_1, c_2]$ infinite outcome till $|[c_1, c_2]_v| \geq |[f(c_1), f(c_2)]_t|$.

The (c_1, ∞) *strategy.* This is of course similar to the above. It is played at a '$[c_1, c_2]$ infinite' stage where we have seen $x_{2,s}$ change since the last such stage. To be (c_1, ∞) believable, an $e^{S(B_s)}$ computation needs to include $[f(c_1), z_1]$ as a successively (if $\#[f(c_1), z_1]$ is below its use), it cannot believe $[z_2, f(c_2)]$ is a successivity and the computation must be consistent with the current information. We preserve such a situation by mapping x_{1,s_1} to z_n with $[z_n, f(c_2)]$, a successivity. For the situation in Diagram 2, if instead of it being a (c_1, c_2)-stage it was a (c_1, ∞) stage we would not get Diagram 2, rather we would get Diagram 4 below.

Again note the addition of $|[c_1, c_2] - 3|$ new points between z_t and f_t and $f(c_2)$, to serve as images to the 'uncovered' elements of $[c_1, c_2]$. As with the previous action, this action can only be injured if its premise is falsified; that is, $[c_1, x_{1,s}]$ is split at some later s stage t. Again we agree that we would play the infinite outcome after stage t until we get $[c_1, c_2]_v \geq |[f(c_1), f(c_2)]_t|$.

The (∞, ∞) *strategy.* This strategy is the most involved of all. A string σ (potentially $S(B)$) is (∞, ∞) correct if

(i) for all z_i, z_j if $\#(z_i, z_j) < u(e^\sigma, x)$ then $\sigma(\#(z_i, z_j)) = 1$ iff $(z_i, z_j) \in S(B_s)$,

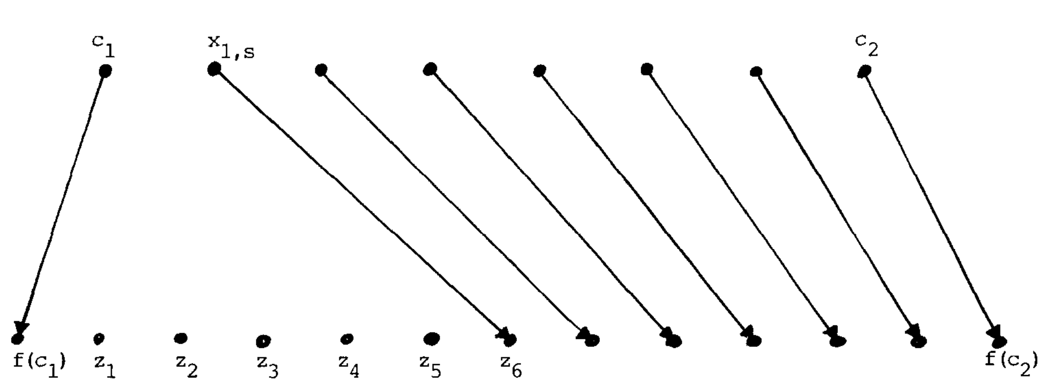

Diagram 4.

(ii) if $\#(f(c_1), z_i) < u(e^\sigma; x)$ then $\sigma(\#(f(c_1), z_i)) = 0$, and
(iii) if $\#(z_i, f(c_2)) < u(e^\sigma; x)$ then $\sigma(\#(z_i, f(c_2))) = 0$.

The idea is that we try to preserve σ by mapping the block $[z_1, z_n]$ to the least candidate for a successivity in $[c_1, c_2]$ consistent with all the outcomes so far. After all, since we are not playing outcome d it must be that for at least some of the $S(A_s)$ successivities are still in test(c_1, c_2, d, s). For a typical situation, in Diagram 5 below, we will indicate by $[d_i, d_{i+1}]$ the possible successivities of $[c_1, c_2]$ if it is the case that test$(c_1, c_2, d, s) = \lim_s$ test(c_1, c_2, d, s). That is, some real successivity is already present. The action will be to map the whole of $[z_1, z_n]$ to the least $[d_i, d_{i+1}]$. In Diagram 5, we suppose $\#[d_i, d_{i+1}] < \#[d_{i+1}, d_{i+2}]$, etc. Again we need to add new points to get consistency.

The situation above can only be injured if we find out $[d_1, d_2]$ is not a successivity. Suppose this occurs at stage t, that is $[d_1, d_2]$ is split in A_t. If we don't wish to play outcome d, it must be the case that at stage t, either $[d_3, d_4]$ or $[d_4, d_5]$ is still a successivity of A_t. Suppose for a typical situation $[d_3, d_4]$ has been destroyed also so that only $[d_4, d_5]$ remains. For simplicity, let $y_{2,t}$ denote the current predecessor of $f(c_2)$, and let $y_{1,t}$ denote the current successor of $f(c_1)$. The action is the same as the one for Diagram 5 except we can use $y_{1,t}$ in place of t_1 and $y_{2,t}$ and d_5 to a $y_{3,t}$ add new points to the rest, and attempt to preserve the block $[y_{1,t}, y_{2,t}]$. Diagram 6 below typifies this situation.

Finally, if we discover that all of the members of test(c_1, c_2, d, s) are not successivities in A (so in our situation $[d_4, d_5]$ gets split), we will play the outcome d. Again we wait till $|[c_1, c_2]_v| > |[f(c_1), f(c_2)]_v| = |[f(c_1), f(c_2)_t]|$.

The general R_{2e} strategy. The above discussion is for a single pair c_1, c_2. In general, R_e will need to be believe a number of δ-fixed points, $c_{1,s}, \ldots, c_{n(e),s}$. It will need to play the appropriate $[c_i, c_{i+1}]$ strategy for each interval, and similarly need strategies to work in the ends. For simplicity we suppose that A has end points p_1 and p_2 so that $c_{1,s} = p_1$ and $c_{n(e),s} = p_2$, at all stages. The guesses will be represented as n-tuples of the form (s, i, μ) corresponding to the belief at stage s, $c_{1,s}, \ldots, c_{n(e),s}$ stabilized (i.e., at all δ-stages $> s$, $c_{i,s} = c_{i,t}$ and $n(e)(s) = n(e)(t)$) and in the interval $[c_i, c_{i+1}]$ the outcome is $\mu \in \{f, (c_1, c_2), (c_1, \infty), (\infty, c_2), (\infty, \infty), d\}$.

The R_{2e+1} strategy. Presented with the above, R_{2e+1} will request that $f_\delta(a_i)$ map to the least element in (or added to) B_s consistent with the higher priority requests. Again, this can cause some interval $|[a, b]_s|$ to no longer match $|[f(a), f(b)]_s$. This is only a worry if $|[a, b]_s| < |(f(a), f(b)]_s|$ so that new elements have been added. This causes no grief since either $[a, b]$ is a finite block so that the Vaught–Remmel theorem applies, or eventually enough points will appear in $[a, b]$ to cover things.

We now turn to the formal details. The priority tree T is generated in stages by

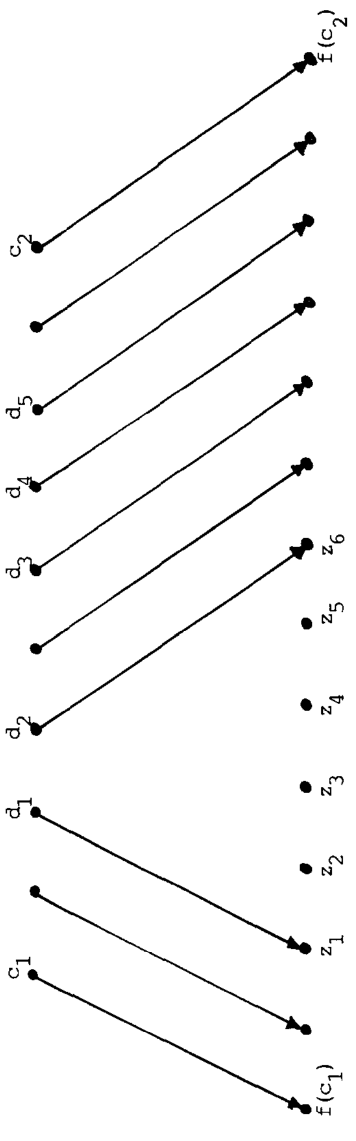

Diagram 5.

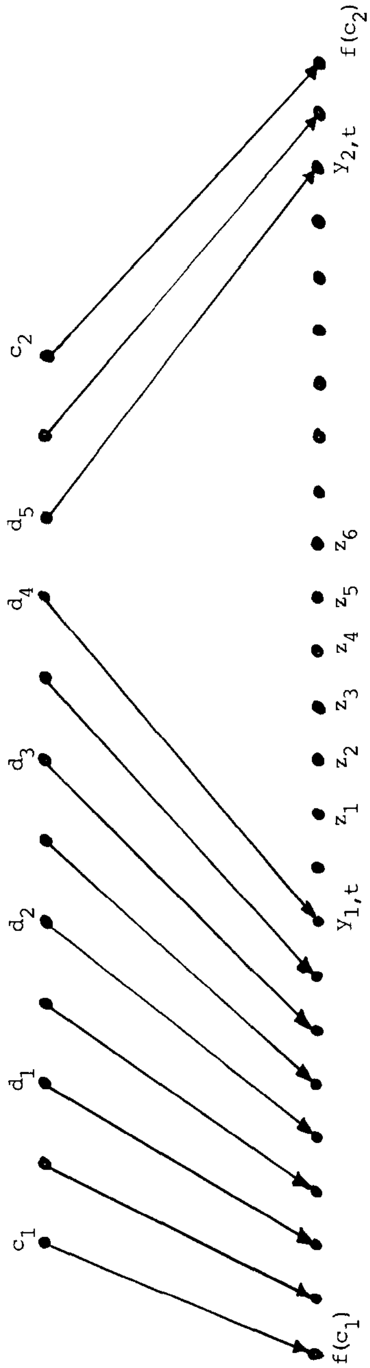

Diagram 6.

associating the R_j on paths in order. Suppose we have generated T on a string σ and R_{2e+1} is the requirement of highest priority not as yet dealt with. For convenience we shall add extra 'layers' to the outcome. First for each canonical finite set D_x and for each $s \in N$ we wish an outcome $\langle s, D_x \rangle$ of σ. We order these lexicographically. The intention here is that s is the stage where we are never left of σ again, all higher priority actions are known, and D_x codes the fixed points we must respect. Moreover, we only consider D_x with the property that $i, j \in D_x$, $i<j$ iff $q_i < q_j$ where q_k denotes the kth rational. To each such $\sigma^\frown\langle s, D_x\rangle$ we associate the outcomes of the form $\tau = \langle \tau_1, \ldots, \tau_{|D_x|+1}\rangle$ where $\tau_i \in \{\infty, f\}$ and τ_1, $\tau_{|D_x|+1} = \infty$, which are ordered lexicographically with $\infty <_{\mathrm{L}} f$. An outcome τ with $\tau_i = \infty$ is meant to encode the belief that $\|[q_{i-1}, q_i]\| = \infty$.

Finally, we expand each outcome τ into its appropriate suboutcomes. So for each τ_i in τ with $\tau_i = \infty$, we have outcomes $\{d, (\infty, \infty), (\infty, c_2), (c_1, \infty), (c_1, c_2)\}$ ordered from left to right, and so to each string τ we associate by lexicographical ordering outcomes $\mu = \langle \mu_1, \ldots, \mu_{|D_x|+1}\rangle$ such that $\mu_i = f$ iff $\tau_i = f$ and $\mu_i \in \{d, (\infty, \infty), (\infty, c_2), (c_1, \infty), (c_1, c_2)\}$ otherwise. This is where we begin to play R_{2e+1}. We assign R_{2e} to each string of the form $\eta = \sigma^\frown\langle s, D_x\rangle^\frown\tau^\frown\mu$ and write $e(\eta) = 2e+1$.

To each stage η we associate R_{2e}'s outcomes i with $i \in \{0, 1\}$. We associate R_{2e+1} with the outcomes of R_{2e}, and hence to a string of the form $\gamma = \sigma^\frown\langle s, D_x\rangle^\frown\tau^\frown\mu^\frown i$. The outcomes of R_{2e+1} are $\mathbb{Q} = \{q_i : i \in \omega\}$ representing the possible choices for $f(a_e)$. We assume $a_{-1} = 0$ and $f: 0 \to 0$ so $b_{-1} = 0$.

Construction

Stage s. We perform the following substages t for $0 \leq t \leq s+1$. We refer to substage t of stage $s+1$ as *stage* (s, t). We append a subscript t to a parameter to indicate its value at the end of stage (s, t). As with all other stages, we will define $f(a_0) = b_0$ $(= a_0)$, as without loss of generality, we can assume $0 < a_0$ so we will be concerned with test sets involving $c_1 = 0$, and $c_2 = a_0$. At substage 0 of stage 0, we set $\mathrm{test}(c_1, c_2, d, 0) = \mathrm{limit}(c_1, 0) = \mathrm{limit}(0, c_2) = \{c_1, c_2\}$. At stages $s > 0$ these sets will already be defined at a previous stage. Let $\mathrm{fixed}(\lambda, s) = \{c_1, c_2\}$.

Substage t

Case 1. We are considering a string σ with $|\sigma| = 0$ or σ is an outcome of some R_{2j+1}. Also we will have defined a set $\mathrm{fixed}(\sigma, s)$, denoting the fixed points generated by the above. Let $\mathrm{fixed}(\sigma, s) = \{c_1, \ldots, c_2\}$ in the A-coding, and let D_x be the corresponding finite set coding $\mathrm{fixed}(\sigma, s)$. Let $t \leq s$ be the least σ-stage such that $\mathrm{fixed}(\sigma, s) = \mathrm{fixed}(\sigma, u)$ for all u with $t \leq u \leq s$. Declare s to be a $\sigma^\frown\langle t, D_x\rangle$-stage.

Now for each subinterval $[c_i, c_{i+1}]$ see if $|[c_i, c_{i+1}]_s| > |[c_i, c_{i+1}]_q|$ where q is the largest σ-stage $\leq s$ with $q \geq t$. Let $\tau = \tau_1, \ldots, \tau_{k+1}$ be the string with $\tau_i \in \{f, \infty\}$, $\tau_1 = \tau_{k+1} = \infty$ (we can assume that both $(\infty, c_1]_s$ and $[c_k, \infty)_s$ have increased in size) and $\tau = \infty$ iff $|[c_i, c_{i+1}]_s| > |[c_i, c_{i+1}]_q|$, otherwise. Declare s to be a $\sigma^\frown\langle t, D_x\rangle^\frown\tau$-stage. Now for each τ_i with $\tau_i = \infty$, we need to determine the appropriate outcome.

For each subinterval of the form $\tau_i = [c_i, c_{i+1}]$ adopt the case below.

Case 1. Each $[d_i, d_{i+1}]$ in test$(c_1, c_2, d, s-1)$ is split in A_s.

Action. Then reset test(c_1, c_2, d, s) to be $\{[q_1, q_2], \ldots, [q_p, q_{p+1}]\}$ where $[x_1, c_2]_s = \{q_1, \ldots, q_{p+1}\}$ in A-order of magnitude. In this case declare that $\mu_i = d$.

Case 2. Not Case 1, and both limit(c_i, s) and limit(s, c_{i+1}) have been reset since the last $\sigma^\frown\langle t, D_x\rangle^\frown\tau$-stage.

Action. Declare that $\mu_i = (\infty, \infty)$. Initialize limit(c_i, s), limit(s, c_{i+1}) to their current apparent values (i.e., limit$(c_i, s) = \{[c_1, d_2]\}$ where d_2 is the successor of c_i in A_s, for instance).

Case 3. Neither Case 1 nor Case 2 apply, and limit(c_i, s) has changed since the last $\sigma^\frown\langle t, D_x\rangle^\frown\tau$-stage.

Action. Declare $\mu_i = (\infty, c_{i+1})$ and initialize limit(c_i, s) to its current apparent value.

Case 4. As with Case 3, except for (c_i, ∞).

Case 5. Otherwise. Declare that $\mu_i = (c_i, c_{i+1})$.

Now we can similarly deal with intervals of the form $(\infty, c_1]$ and $[c_k, \infty)$, except they can only have μ_i of the form d, (∞, ∞), (∞, c_1) or (c_k, ∞). With this modification we generate μ_1 and μ_{k+1}. We then declare that s is a $\sigma' = \sigma^\frown\langle t, D_x\rangle^\frown\tau^\frown\mu$-stage where $\mu = \mu_1, \ldots, \mu_{k+1}$ where $\mu_i = f$ iff $\tau_i = f$ and μ_i is generated by the above if $\tau_i = \infty$. Initialize all γ for $\gamma \not\leq_{\mathrm{L}} \sigma^+$, where $\leq_{\mathrm{L}}$ denotes the standard lexicographical ordering.

Now we deal with R_{2e+1} where $e(\sigma^+) = 2e+1$. We assume first that R_{2e+1} is not as yet declared satisfied. If it does not yet have a follower with guess σ^+, give it one, say $\langle\sigma^+, s\rangle$, and declare that s is a $\sigma^{+\frown}0$-stage. Otherwise we can assume it has a follower $x = x(\sigma^+, s)$ not yet in D_s. As with discussion preceding the construction, we can decide if a computation

$$e_s^\gamma(x) = 0$$

is compatible with γ being an initial segment of $S(B)$ according to the guess σ^+. Let $u(\gamma)$ be the use of such a computation. Then for all $z < u(\gamma)$ we need that if z is the code a pair $\langle p, q\rangle$ then (i) and (ii) below hold.

(i) $[p, q]$ not a successivity in B_s iff $\gamma(z) = 0$.

(ii) If $[p, q]$ is a successivity in B_s then find i with $[p, q] \subseteq (\infty, c_i]_s$, or $[p, q] \subseteq [c_i, c_{i+1}]$ or $[p, q] \subseteq [c_k, \infty)$.

Without loss of generality we suppose $[p, q] \subseteq [c_i, c_{i+1}]$. We ask that γ is consistent with μ_i, as with the discussion before the construction. So we ask that $\mu_i \neq d$. If $\mu_i = (\infty, \infty)$ then $p \neq c_i$ and $q \neq c_{i+1}$. If $\mu_i = (\infty, c_2)$ then $p \neq c_i$. And if $\mu_i = (c_1, \infty)$ then $q \neq c_{i+1}$.

If we see a string γ coding successivities of B_s compatible with σ^+, then we declare that R_{2e+1} is *satisfied* (at σ^+), and declare that s is a $\sigma^{+\frown}1$-stage. We enumerate x into D. Now we will restrain γ to preserve this win. Again we follow the technique of the basic module, now on each interval $[c_i, c_{i+1}]$, $(\infty, c_i]$ and $[c_{i+1}, \infty)$. Again we only look at $[c_i, c_{i+1}]$.

If $\mu_i = d$, then there is nothing to restrain for γ. If $\mu_i = (\infty, \infty)$, define $\text{restrain}(\sigma^+, [c_i, c_{i+1}], s) = [z_1, z_n]$ where $f_\sigma([c_i, c_{i+1}]_s) = \{f_\sigma(c_i), z_1, \ldots, z_n, f_\sigma(c_{i+1})\}$ in B-order. Find the least j such that $[d_j, d_{j+1}] \in \text{test}(\sigma, c_i, c_{i+1}, s)$ and $[d_j, d_{j+1}]$ is a successivity in A_s. Define $f_{\sigma^+,s}(d_j) = z_1$ and $f_{\sigma^+,s}(d_{j+1}) = z_n$. Put d_j and d_{j+1} into $\text{fixed}(\sigma^+, s)$. The cases (∞, c_2), (c_1, ∞), (c_1, c_2) are treated similarly.

Now for the case that R_{2e+1} has been declared satisfied at guess σ^+, for each i with $\mu_i = (\infty, \infty)$ find the least j with $[d_j, d_{j+1}] \in \text{test}(\sigma, c_i, c_{i+1}, s)$ and with $[d_j, d_{j+1}]$ still a successivity in A_s. Define $f_{\sigma^+,s}$ as above. (This may or may not change f_{σ^+} since the last σ^+-stage.)

If none of the above apply declare that s is a $\sigma^{+\frown}0$-stage.

Case 2. σ is devoted to R_{2e}. We wish to define $f_{\sigma,s}(a_e)$. Let $\text{fixed}(\sigma, s) = \{c_1, \ldots, c_k\}$ and let $a_e \in [c_i, c_{i+1}]_s$ without loss of generality, end points being treated similarly. If $a_e \in \{c_1, \ldots, c_k\}$ we need do nothing. If $a_e \notin \{c_1, \ldots, c_k\}$ find the least b_j, if any, with $b_j \in [f_{\sigma,s}(c_i), f_{\sigma,s}(c_{i+1})]$ and $b_j \notin \text{restrain}(\tau, s)$ for any $\tau \leqslant_L \sigma$. If b_j exists, define $f_{\sigma,s} = b_j$ and declare that s is a $\sigma^\frown b_j$-stage. If no such b_j exists find a new rational b_k with Gödel numbering bigger than s, such that b_k is consistent with $f_{\sigma,s}(c_1), \ldots, f_{\sigma,s}(c_k)$ as well as respecting all of $\text{restrain}(\tau, s)$ for $\tau \leqslant_L \sigma$. Define $f_{\sigma,s}(a_e) = b_k$.

Now at substage $t = s$, initialize all τ with $\tau \nleqslant_L \sigma$ and where s is a σ-stage.
End of construction

Verification. Let β be the true path. Let $\beta(e)$ denote the node on β devoted to R_e. We prove by simultaneous induction that R_{2e+1} only receives attention finitely often at β-stages, and hence $\lim_s \text{restrain}(\tau, s) = r(\sigma)$ exists for all $\sigma \subseteq \beta$, and $\lim_s f_{\beta(2e),s}(a_e) = f_{\beta(2e)}(a_e)$ exists, and has the desired properties.

First we argue for the R_{2e+1}. Let s_0 be a stage where we are never again left of $\beta(2e+1)$ and all the R_j for $j < 2e+1$ have ceased activity. Then $\text{fixed}(\beta(2e), s)$ has come to a limit. It is easy to see that it suffices to show that if R_{2e+1} receives attention after stage s_0 at $\beta(2e+1)$, this action will succeed. If R_{2e+1} does not again receive attention, no string γ is compatible with the (true) guess $\beta(2e+1)$ exists with $e^\gamma(x)\downarrow = 0$, and hence R_{2e+1} is met by nonagreement. Suppose that R_{2e+1} receives attention via γ. It suffices to argue that for each interval $[c_i, c_{i+1}]$ in $\text{fixed}(\beta(2e+1), s) = \text{fixed}(\beta(2e+1))$ we succeed with $\text{restraint}(\beta(2e+1), s)$.

Now this restraint is successful at all δ-stages $s > s_0$ where $\delta \subseteq \beta$ by choice of s_0 (for the R_j for $j < 2e+1$) and by the fact that R_{2k} of lower priority than R_{2e+1} must respect restraint($\beta(2e+1), s$), due to the fact that the end points are in the fixed point set of $(\beta(2e+1), q)$ for *all* stages $q \geq s$. We are now again left of $\beta(2e+1)$ by choice of s_0.

So suppose we move right of $\beta(2e+1)$ at stage $t \geq s$. The only possible injury to the restraint set is due to the action of an R_{2n+1}, while such a requirement can define f_δ to differ from $f_{\beta(2e+1)}$, we claim that restrain(δ, t) $\supseteq$ restrain($\beta(2e+1), s$).

To see this we need some case analysis. Let $\beta(2e+1) = \sigma^\frown\langle s, D_x\rangle^\frown\tau^\frown\mu$. We argue locally on μ_i for each i. If $\mu_i = d$, $[c_i, c_{i+1}]$ has no elements of restrain($\beta(2e+1), s$). If $\mu_i = (\infty, \infty)$, since we are never again left of μ_i one of the apparent successivities at stage s is a real successivity. Since we adjust the map to the apparent successivity at each stage, we can presume that in fact restrain(μ_i, s) is the image of a real successivity of A, say $[d_1, d_2]$. This implies that the restrain(μ_i, s) is well defined and is never reset. Since the placement of new points b_j must respect higher priority restraints, it follows that no b_j is added between the least and greatest element of restrain(μ_j, s). The other cases are similar.

Finally, we need to argue that f works. Clearly $\lim_s f_{\beta,s}(a_e) = f_\beta(a_e)$ exists for all a_e. Furthermore the mapping is isotone and 1–1 by construction. Suppose $b_j \in \beta$ and $b_j \notin \operatorname{ra} f_\beta$. We claim that either there is a $\sigma \subseteq \beta$ such that for almost all stages $b_j \in$ restrain(σ, s), or for some k, $f_\beta(a_k) = b_j$, or $b_j \in f_\beta[c_i, c_{i+1}]$ and $|(c_i, c_{i+1}]| < \infty$. Suppose that this claim is valid for all b_d with $d < J$ and s_0 be the stage witnessing the truth of the claim for all $d < j$. That is for all such b_d, either $b_d \in$ restrain($\sigma(d)$) $=$ restrain($\sigma(d), s$) or $f_{\beta,s}(a_{k/d})) = b_d$ henceforth. Let $\delta \subseteq \beta$ be the guess by which all such b_d are resolved. We may assume $b_j \in B_s$. Let fixed(δ, s) $= \{c_1, \ldots, c_k\} =$ fixed(δ). Now for some i, $b_j \in f_\delta([c_i, c_{i+1}])$. If it is the case that $|[c_i, c_{i+1}]) < \infty$, there is nothing to prove. So suppose otherwise. Let a_m be the least element to enter $[c_i, c_{i+1}]$ after stage s. Now let t be the first δ- and ρ-stage with $\rho \subseteq \beta$ and ρ denoted by R_{2m}. We might as well suppose that $\delta \subseteq \rho$. Let $t_1 > t$ be the least ρ-stage where all of the restrain(τ, t_1) for $\tau \leq_{\mathrm{L}} \rho$ are now fixed, we are never again left of ρ, and all fixed(τ, t_1) are final. Now if we define $f_{\rho,t}(a_m) = b_j$ then we are done since this map will be refined to all ρ-stages $> t_1$. The only reason we would not define $f_{\rho,t}(a_m) = b_j$ would be because $b_j \in$ restrain(τ, t_1) for some $\tau \leq_{\mathrm{L}} \rho$. But then it follows that since restrain(τ, f_1) $=$ restrain(τ), for some η with $\eta \subseteq \rho$ and $\tau \leq_{\mathrm{L}} \eta$, it will be the case that $b_j \in$ restrain(η). By construction this means that if restrain(η) $= [z_1, z_2]$, then $[f_\eta^{-1}(z_1), f^{-1}\eta(z_2)]$ is a successivity in A. This concludes the proof.

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