Kobayashi compressibility [*]

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Abstract.   Kobayashi [Kob81] introduced a uniform notion of compressibility of infinite binary sequences X in terms of relative Turing computations with sub-linear use of the oracle. Given $f : \mathbb{N} \to \mathbb{N}$ we say that X is f-compressible if there exists Y such that for each n we can uniformly compute $X \restriction_n$ from oracle $Y \restriction_{f(n)}$. Kobayashi compressibility has remained a relatively obscure notion, with the exception of some work on resource bounded Kolmogorov complexity. The main goal of this note is to show that it is relevant to a number of topics in current research on algorithmic randomness.

We prove that Kobayashi compressibility can be used in order to define Martin-Löf randomness, a strong version of finite randomness and Kurtz randomness, strictly in terms of Turing reductions. Moreover these randomness notions naturally correspond to Turing reducibility, weak truth-table reducibility and truth-table reducibility respectively. Finally we discuss Kobayashi's main result from [Kob81] regarding the compressibility of computably enumerable sets, and provide additional related original results.

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1 Introduction

The compressibility of a finite binary program σ is defined in terms of the shortest program that can generate σ. This is the idea behind the theory of Kolmogorov complexity C of strings. For example, if $c \in \mathbb{N}$ then σ is c-incompressible if $C(\sigma) \geq |\sigma| - c$, and similar definitions are used with respect to the prefix-free complexity K, where the underlying universal machine is prefix-free. This notion of incompressibility has a well-known extension to infinite binary streams X, where we say that X is c-incompressible if $K(X \upharpoonright_n) \geq n - c$ for all n. Then the algorithmic randomness of X is often identified with the property that X is c-incompressible for some c, and coincides with the notion of Martin-Löf randomness[1]. These concepts are basic in Kolmogorov complexity, and the reader is referred to the standard textbooks [LV97, DH10] for the relevant background.

1.1 Kobayashi compressibility and incompressibility

The reader may observe that the above extension of the definition of compressibility from finite to infinite sequences is nonuniform, in the sense that the compression of the various initial segments of X could be done by different, possibly unrelated short programs. A uniform extension of compressibility from strings to infinite streams would require the individual short programs to be part of a single stream. Kobayashi [Kob81] considered exactly that approach.

Definition 1.1 (Kobayashi [Kob81])**.** *Given $f : \mathbb{N} \to \mathbb{N}$ we say that X is f-compressible if there exists Y such that for each n we can uniformly compute $X \upharpoonright_n$ from oracle $Y \upharpoonright_{f(n)}$.*[2]

Note that, since every real is computable from itself with identity use, Definition 1.1 only makes sense if $f(n)$ occasionally dips well below n. This feature contrasts a standard caveat that is often assumed in computability theory for convenience, that the oracle-use in relative computations is strictly increasing. Kobayashi did not necessarily require that f is computable in this definition, but added effectivity requirements in the statements of his results. We formulate the corresponding notion of incompressibility of a real X based on Definition 1.1 as follows.

Definition 1.2 (Kobayashi incompressibility)**.** *We say that a real X is Kobayashi incompressible if it is not f-compressible for any function f such that $n - f(n)$ is unbounded.*

Note that every set X is $(n - c)$-compressible for every constant c. Indeed, given c one can consider Y such that $X = X \upharpoonright_c * Y$, and by hardwiring $X \upharpoonright_c$ into a Turing machine we can compute X from Y with oracle-use $n - c$.

Kobayashi showed that the class of the incompressible streams X of Definition 1.2 has measure 1. We will see in the following that, in fact, this definition is equivalent to Martin-Löf randomness. Furthermore, if Turing computability in this definition is replaced with stronger reducibilities, then we get

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[1]This is the first robust and most accepted definition of algorithmic randomness and was originally introduced by Martin-Löf [ML66] based on effective statistical tests.

[2]The reader who is familiar with monotone complexity from Levin in [Lev71, Lev73] (also discussed in [DH10, Section 3.15]) will note that if X is f-compressible for a computable function f, then f is an upper bound on the monotone complexity of X.

alternative definitions of Kurtz randomness[3] and a strong version of computably bounded randomness[4] which we call granular randomness, respectively. We state these results in Section 1.3, deferring their proofs in latter sections. It is interesting to note that these alternative definitions do not involve measure or prefix-free machines, so they are unique in that they only use notions from classical computability theory. It is curious that Kobayashi's simple and natural notion of compressibility has remained rather obscure, and does not even feature in the encyclopedic books on Kolmogorov complexity and computability [LV97, DH10, Nie09].[5]

1.2 Previous work on Kobayashi compressibility

An adaptation Kobayashi's compressibility notion for resource-bounded computations was considered in Balcázar, Gavaldà, and Hermo [BGH94, BGH96], where it was shown that for logarithmic initial segment complexity (i.e. when the plain or prefix-free Kolmogorov complexity of the sequence is $\mathbf{O}(\log n)$), the uniform and nonuniform approaches coincide, both in the resource bounded and resource unbounded case. In particular, they showed that[6]

> if X has has initial segment complexity $\mathbf{O}(\log n)$ then there exists some Y that computes X with oracle-use $\mathbf{O}(\log n)$.

Another observation from [BGH94, BGH96] is that for constant oracle-use bounds and polynomial time, Kobayashi's notion coincides with the polynomial computable functions and also with the corresponding resource-bounded version in terms of initial segment complexity relative to the length of the initial segment, which was studied by Loveland [Lov69].

Such uniform notions of compressibility for infinite streams based on Kobayashi's report were later used by Balcázar, Gavaldà and Siegelmann in [BGS06] in order to give a characterization of the computational power of recurrent neural networks in terms of the Kolmogorov complexity of their weights. A hierarchy theorem regarding resource-bounded compressibility notions based on functions f of increasing growth rates is also included in [BGS06].

The notion of relative computations with occasionally sub-linear oracle-use that is behind Kobayashi's notion of compressibility also initiated a study of strong reducibilities in a series of three never-before-cited papers by Habart [Hab90, Hab91, Hab92]. From these papers, [Hab90] is directly related to algorithmic randomness notions. Given a computable $f : \mathbb{N} \to \mathbb{N}$ let $\mathcal{S}(f)$ be the class of reals X which are f-compressible according to Definition 1.1. Given a subclass C of the class ω^ω of functions from the natural numbers to the natural numbers, Habart defined X to be C-incompressible (or incompressible with respect to C) if

$$\forall f \in C, \ [X \in \mathcal{S}(f) \Rightarrow \mathcal{S}(f) = 2^\omega] \tag{1.2.1}$$

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[3]originally from Kurtz [Kur81] and further studied in Downey Griffiths and Reid [DGR04].

[4]introduced and studied by Brodhead, Downey and Ng [BDN12].

[5]Of the two citations to Kobayashi's work in [LV97] one is about a somewhat known result regarding the structure of one-tape nondeterministic Turing machine time hierarchy and the other is [Kob93]. Incidentally, the results in the latter paper were independently reproved by Becher, Figueira, Grigorieff and Miller [BFGM06] (along with other original results).

[6]The reader may recall that $\mathbf{O}(\log n)$ initial segment complexity means the same (modulo additive constants) for plain or prefix-free Kolmogorov complexity, since the latter is at most two times the plain complexity.

where 2^ω is the class of all infinite binary sequences. Similarly, X is *C-compressible* if it is not *C*-incompressible, i.e. if there exists $f \in C$ such that $X \in S(f)$ and there are f-incompressible reals. Habart then shows that

> Martin-Löf randomness is equivalent to incompressibility
> with respect to all non-decreasing functions in ω^ω.

Moreover Habart shows that the following conditions are equivalent for any real X:

(a) X is compressible with respect to the computable functions in ω^ω;

(b) X is compressible with respect to the nondecreasing computable functions in ω^ω;

(c) there exists a computable function g such that $K(X \restriction_{g(n)}) \le g(n) - n$ for all n.

The follow-up papers [Hab91, Hab92] explore variations of the above notion of compressibility and, amongst other results, give a characterization of bi-immunity in terms of compressibility.

The main result from [Kob81] is the following characterization of the compressibility of computably enumerable (c.e. from now on) sets in terms of asymptotic bounds on the oracle use in relative computations.

Theorem 1.3 (Kobayashi [Kob81]). *If f is a non-decreasing computable function then the following are equivalent:*

(a) $\sum_n 2^{-f(n)}$ *is finite;*

(b) *every c.e. set X is computable by some set Y with oracle-use bounded above by f.*

This is an elegant characterization of the compressibility of c.e. sets. Other results on this topic, such as the work in [Bar68, Kum96, BL13, BHLM13], tend to focus on the Kolmogorov complexity of the initial segments of c.e. sets. In fact, Theorem 1.3 remains true if we add a third clause saying 'every c.e. set X is computable by Ω with oracle-use bounded above by f' where Ω is Chaitin's halting probability. We give a short proof of this slightly enhanced result in Section 3.1.

In the last decade, a number of vaguely related results in the spirt of Theorem 1.3 have emerged in the literature. Tadaki [Tad09] considered the problem of how many bits of Ω are needed in order to compute the domain of the universal prefix-free machine U up to the strings of length n. He showed that, if we restrict the question to computable use-functions, then the answer is given by the computable functions of the type $n - f(n)$ such that $\sum_n 2^{-f(n)}$ is finite. Similar asymptotic conditions on the oracle use in relative computations between c.e. reals were obtained in [BFLP16].

A typical case in Theorem 1.3 is for logarithmic bounds. We get that if $\epsilon > 1$ then every c.e. set is $(\epsilon \cdot \log n)$-compressible while there are c.e. sets that are not $(\log n)$-compressible. A result with the same flavor was recently presented in [BLP16], where it was shown that halting probabilities of universal prefix-free machines are computable from eachother with use $n + \epsilon \cdot \log n$ for any $\epsilon > 1$, but this statement is no longer true for $\epsilon = 1$.

1.3 Our results

There are two types of results that we present, both of which are related to Kobayashi's report [Kob81]. First, in Section 1.3.1 we consider versions of Kobayashi incompressibility and relate them to known randomness notions. Second, in Section 1.3.2 we consider some results related to the initial segment complexity of c.e. sets and Theorem 1.3.

1.3.1 Kobayashi incompressibility and reductions

Our first result is that Kobayashi incompressibility coincides with Martin-Löf randomness. This fact has also been noticed by Laurent Bienvenu (personal communication—unpublished). Recall that a set of strings U can be viewed as the set of reals that have a prefix in U. In this way, the measure $\mu(U)$ is the Lebesgue measure of the corresponding open set of reals. Also recall that a Martin-Löf test is a uniformly c.e. sequence of sets (U_i) with $\mu(U_i) < 2^{-i}$ for all i, and a real X is Martin-Löf random if it is not in $\cap_i U_i$ for any Martin-Löf test (U_i).

Theorem 1.4 (Kobayashi incompressibility and Turing reductions). *The following are equivalent:*

(a) X is Martin-Löf random;

(b) for every Y such that $X \leq_T Y$ the Y-use in any such computation of $X \upharpoonright_n$ is bounded below by $n - c$ for some constant c and all n.

Next, we consider the notion of Kobayashi compressibility, as it was discussed in Section 1.1, but with respect to strong reducibilities like truth-table and weak truth-table reducibility. Before we state our results, a few remarks regarding strong reducibilities with sub-linear oracle use are in order. There is a uniform and a nonuniform way to define truth-table and weak truth-table reductions, when there is a strong emphasis on the oracle-use of the associated computations. The uniform way corresponds to the case where the oracle-bound is oblivious to the oracle used in the reduction. For example, $X \leq_{tt} Y$ with oracle-use f means that there exists a total Turing functional Φ with uniform oracle-use f on any oracle X, such that $\Phi^Y = X$. The non-uniform definition would be to say that there exists a total Turing functional Φ such that $\Phi^Y = X$ and the oracle-use on Y is bounded by f. Similar formulations apply to weak truth-table reductions. It is not hard to see that the two formulations are equivalent when f is computable, since we can modify Φ to behave in a trivial way on oracles that do not respect the given oracle-use bound. In our presentation we follow the uniform formulation.

With this clarification out of the way, we can modify Kobayashi's notion of compressibility by replacing Turing reductions for strong reductions. For example, given a computable $f : \mathbb{N} \to \mathbb{N}$ we can say that X is truth-table f-compressible if there exists Y which truth-table computes X with oracle-use bounded above by f. Similar definitions apply with respect to weak truth-table reducibility. It is now natural to ask whether these restricted notions of incompressibility characterize known variants of algorithmic randomness. It turns out that this is indeed the case, and for the truth-table variant the corresponding notion is Kurtz randomness, which was originally introduced in [Kur81]. A Martin-Löf test (U_i) is a *Kurtz test* if U_i if finite for each i. A real is *Kurtz random* if it is not in $\cap_i U_i$ for any Kurtz test (U_i).

Theorem 1.5 (Kobayashi incompressibility and truth-table reductions). *The following are equivalent:*

(a) X is Kurtz random;

(b) for every Y such that $X \leq_{tt} Y$ the Y-use in any such computation of $X \upharpoonright_n$ is bounded below by n − c for some constant c and all n.

Downey, Griffiths and Reid [DGR04] gave alternative characterizations of Kurtz randomness in terms of tests and initial segment complexity. These characterizations can be used to give an alternative proof of 1.5, as we note in the following.

If we consider Theorem 1.4 with respect to weak truth-table reducibility, the corresponding randomness notion is what we call granular randomness in Theorem 1.6, which expresses it as a restricted version of Martin-Löf randomness in terms of tests, martingales, or prefix-free complexity. Given a function g, we say that a Martin-Löf test (V_i) is g-granular if for each i each string in V_i has length at most $g(i)$. Moreover we say that a Martin-Löf test is computably granular if it is g-granular for some computable non-decreasing function g.

Theorem 1.6 (Granular randomness). *The following are equivalent for a real X:*

(a) $X \in \cap_i V_i$ for a computably granular Martin-Löf test (V_i);

(b) $M(X \upharpoonright_{g(n)}) \geq n$ for all n, a c.e. (super)martingale M and a computable function g;

(c) $K(X \upharpoonright_{g(i)}) \leq g(i) − i$ for a computable function g.

A real which does not have these properties is called granularly random.

Note that, given a granular test (V_i), without loss of generality we may assume that g is increasing and that for each i, all strings in V_i have length $g(i)$. Moreover since $\mu(V_i) \leq 2^{-i}$ for each i, we may also assume that $|V_i| \leq 2^{g(i)-i}$ for each i. This means that granular randomness is weaker[7] than all versions of finite randomness that was introduced and studied by Brodhead, Downey and Ng in [BDN12]. On the other hand, granular randomness is stronger[8] than Kurtz randomness.

Theorem 1.7 (Kobayashi incompressibility and bounded oracle-use). *The following are equivalent:*

(a) X is granularly random;

(b) for every Y such that $X \leq_{wtt} Y$ the Y-use in any such computation of $X \upharpoonright_n$ is bounded below by n − c for some constant c and all n.

This last result is related to a result from Habart [Hab90], which asserts that for each real X, property (c) of Theorem 1.6 is equivalent to X being $\mathcal{R}$-compressible in the sense of definition (1.2.1), where $\mathcal{R}$ here denotes the class of computable functions.

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[7]in fact, strictly weaker. By a diagonalizing against all granular Martin-Löf tests it is a simple exercise to construct a granularly random real which is not random with respect to any of the finite randomness notions of [BDN12].

[8]in fact, strictly stronger. By a diagonalizing against all Kurtz tests it is a simple exercise to show that there are Kurtz random reals which are not granularly random.

1.3.2 Oracle-use in computations of c.e. sets and c.e. reals

The number Ω was introduced by Chaitin in [Cha75] as the halting probability of a universal prefix-free machine, who also showed that it is Martin-Löf random. Clearly Ω depends on the underlying universal prefix-free machine, but the cumulative work of Solovay [Sol75], Calude, Hertling, Khoussainov and Wang [CHKW01] and Kučera and Slaman [KS01] showed that these numbers are exactly the left-c.e. Martin-Löf random numbers. Finally we point out the following fact regarding left-c.e. reals with highly complex initial segment. As the universal halting probability, Ω computes all left-c.e. reals including the halting problem itself. The following theorem qualifies the latter completeness property and relates to the main result of Kobayashi [Kob81], Theorem 1.3.

Theorem 1.8. *Every left-c.e. real X can be computed from Ω with oracle-use $g(n) = \max_{i \leq n} K(X \upharpoonright_n)$.*[9]

Recall that $K(\sigma) \leq |\sigma| + K(|\sigma|)$ for all strings σ, and that if g is a computable function such that $\sum_n 2^{-g(n)}$ is finite, then $K(n) \leq g(n) + \mathbf{O}(1)$ for all n. Hence Theorem 1.8 implies that any left-c.e. real is computable from Ω with oracle-use $n + g(n)$, where g is any computable non-decreasing function such that $\sum_n 2^{-g(n)}$ is finite. The latter fact was shown in [BFLP16], where it was also shown to be optimal in the sense that if $\sum_n 2^{-g(n)}$ is infinite then there exists a left-c.e. real which is not computable from Ω with with oracle-use $n + g(n)$.

It would be interesting to find in which sense the upper bound of Theorem 1.8 for individual left-c.e. reals X is optimal. This might not be straightforward as computable left-c.e. reals are computable from any Ω with zero oracle-use and Martin-Löf random left-c.e. reals are versions of Ω and are computed from themselves with identify oracle-use. With regard to the latter case, we do know from [BLP16] that given any version of Ω we can find a random left-c.e. real which is not computable from Ω with use $\log n + \mathbf{O}(1)$.

Barzdins [Bar68] showed that the plain complexity of every c.e. set is bounded above by $2 \log n + \mathbf{O}(1)$. Then Kummer [Kum96] showed that this upper bound is optimal, in the sense that there exist c.e. sets A such that for some constant c we have $C(A \upharpoonright_n) > 2 \log n - c$ for infinitely many n. Here we present a corresponding optimal monotone upper bound for the prefix-free complexity of c.e. sets.

Theorem 1.9. *If f is a computable non-decreasing function then the following are equivalent:*

(a) *the prefix-free complexity of every c.e. set is bounded by $f(n) + \log n + \mathbf{O}(1)$;*

(b) *$\sum_n 2^{-f(n)}$ is finite.*

Recall from [DJS96] that a Turing degree $\mathbf{a}$ is array computable if there exists a function which is truth-table reducible to the halting problem, which dominates all functions computable from $\mathbf{a}$. Kummer [Kum96] showed that array non-computable c.e. degrees contain c.e. sets A such that $C(A \upharpoonright_n) > 2 \log n - c$ for some constant c and infinitely many n, while the plain complexity of the c.e. sets of an array computable degree is bounded above by $\log n + g(n) + \mathbf{O}(1)$ for any given non-decreasing unbounded computable function g. This is sometimes known as the Kummer gap theorem (see [DH10, Section 16.1]).

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[9]The reader should not be confused by the fact that we consider non-computable functions as oracle-use. Formally, we may say that X is computable from Y with oracle-use g, if there exists an oracle machine M such that for any n and any real Z extending $Y \upharpoonright_{g(n)}$ on the oracle tape of M we have $M^Z(n) = X(n)$.

The prefix-free version of the Kummer gap theorem says that the upper bound $f(n) + \log n$ of Theorem 1.9 is tight for array non-computable c.e. degrees (in the same sense that $2 \log n$ was tight in the plain complexity case) and that $f(n) + g(n) + \mathbf{O}(1)$ is an upper bound for the prefix-free complexity of array computable c.e. sets (where again, g is any fixed computable unbounded non-decreasing function). We state a representative case of this result, while the interested reader can verify that the proof that we give in Section 3.4 applies to the general case.

Theorem 1.10 (Kummer gap for prefix-free complexity of c.e. sets). *If $\mathbf{a}$ is a c.e. degree then*

(a) *if $\mathbf{a}$ is array non-computable then it contains a c.e. set whose prefix-free complexity is not bounded above by $2 \log n + \log \log n$;*

(b) *if $\mathbf{a}$ is array computable then the prefix-free complexity of all of its c.e. members is bounded above by $\log n + 2 \log \log n + \mathbf{O}(1)$.*

Note that the upper bound for the case of array non-computable degrees is slightly higher than Kummer's $2 \log n$, as it follows from Theorems 1.10 and 1.9. The same is true for the case of array computable degrees, where we Kummer's bound was $\log n + g(n)$ for an arbitrary unbounded computable non-decreasing function g.

2 Characterizations of randomness notions

The arguments we give for Theorems 1.5, 1.5 and 1.7. are based on proving $\neg$(a) $\leftrightarrow$ $\neg$(b) in each case. We see two different ways of proving $\neg$(a) $\leftarrow$ $\neg$(b) in each of these cases. One is to construct Y and the required reduction of X to Y based on the definition of the randomness notion at hand, in terms of tests. The second way is to use the Kolmogorov complexity definition of the corresponding randomness notion. We demonstrate the first methodology in the proof of Theorem 1.5 and demonstrate the second methodology in the proofs of Theorems 1.5 and 1.7.

2.1 Proof of Theorem 1.4

We need to show that the following are equivalent:

(a) X is Martin-Löf random;

(b) for every Y such that $X \leq_T Y$ the Y-use in any such computation of $X \upharpoonright_n$ is bounded below by $n - c$ for some constant c and all n.

It suffices to prove $\neg$(a) $\leftrightarrow$ $\neg$(b). First, assume that X is not Martin-Löf random. Then for every constant c there exists some n such that $K(X \upharpoonright_n) \leq n - c$. Hence for every c there exists some n and a description of $X \upharpoonright_n$ of length at most $n - c$. We can use this fact in order to define a stream Y which computes X with use some function g such that $g(n) \leq n$ for infinitely many n. We define Y by induction, as the concatenation of a suitably crafted sequence of strings (σ_i). Let σ_0 be the empty sequence and let $n_0 = c_0 = 0$. Inductively assume that for some $k > 0$ all $\sigma_i, i < k$ have been defined and belong to the domain of the universal prefix-free machine U. Let c_k be the sum of all $|\sigma_i|, i \leq k$. Let $n_k > n_{k-1}$ be a

number such that $K(X \upharpoonright_{n_k}) \leq n_k - c_{k-1}$ and let σ_k be the shortest description of $X \upharpoonright_{n_k}$. This concludes the definition of (σ_i).

Let $Y = \sigma_0 * \sigma_1 * \cdots$ and note that, since all σ_i are in the domain of U and U is a prefix-free machine, for each k we can uniformly compute $X \upharpoonright_{n_k}$ from $Y \upharpoonright_{c_k}$. We now show by induction that for each k we also have $c_k \leq n_k$. Clearly this holds for $k = 0$. Inductively assume that $k > 0$ and $c_{k-1} \leq n_{k-1}$. Then $c_k = c_{k-1} + K(X \upharpoonright_{n_k}) \leq c_{k-1} + n_k - c_{k-1} = n_k$ which concludes the induction step. Note that since U is prefix-free the oracle Y can compute the sequence (c_i). Therefore for each n the string $X \upharpoonright_n$ can be uniformly computed by Y with use c_k, where k is the least number such that $n \leq n_k$. So assuming that $\neg$(a) we have proved that $\neg$(b).

For the other direction assume that $\neg$(b) and let Y be an oracle such that $X = \Phi(Y)$ for a Turing functional Φ such for each c there exists n such that the Y-use $\phi(X \upharpoonright_n)$ for the computation of $X \upharpoonright_n$ is less than $n - c$. We can view Φ as a c.e. set of tuples $\langle \sigma, \tau \rangle$ which indicate that any stream with prefix σ is Φ-mapped to a string or stream with prefix τ. We enumerate a Martin-Löf test (V_i) as follows. Given k, for each $\langle \sigma, \tau \rangle$ in Φ such that $|\tau| > k + |\sigma|$ we enumerate τ into V_k. By our assumption we have $X \in \cap_i V_i$, so it remains to show that $\mu(V_k) \leq 2^{-k}$ for each k. For a contradiction, suppose that $\mu(V_k) > 2^{-k}$ for some k. Then there exists a finite prefix-free set of strings $\{\tau_i \mid i < t\} \subseteq V_k$ such that

$$\sum_{i<t} 2^{-|\tau_i|} > 2^{-k}.$$

By the way we enumerate V_k and since Φ is a Turing functional, for each $\tau_i, i < t$ there exists some σ_i such that $\langle \sigma_i, \tau_i \rangle \in \Phi, |\sigma_i| < |\tau_i| - k$ and the set $\{\sigma_i \mid i < t\}$ is prefix-free. So

$$\sum_{i<t} 2^{-|\tau_i|} < \sum_{i<t} 2^{-|\sigma_i|-k} \leq 2^{-k} \cdot \sum_{i<t} 2^{-|\sigma_i|} \leq 2^{-k}$$

which contradicts the previous inequality. We can conclude that $\mu(V_k) \leq 2^{-k}$ for each k, which completes the proof that X is not Martin-Löf random.

2.2   Proof of Theorem 1.5

We need to show that the following are equivalent:

(a) X is Kurtz random;

(b) for every Y such that $X \leq_{tt} Y$ the Y-use in any such computation of $X \upharpoonright_n$ is bounded below by $n - c$ for some constant c and all n.

We show $\neg$(a) $\leftrightarrow$ $\neg$(b). Note that for $\neg$(a) $\rightarrow$ $\neg$(b) one can use the characterization of Kurtz randomness in terms of a restricted version of prefix-free complexity from Downey Griffiths and Reid [DGR04] (just as we used such a characterization of Martin-Löf randomness in Section 2.1). Instead, here we use the formulation of Kurtz randomness in terms of tests, originally by Wang [Wan96].

▶ *Proof that $\neg$(a) implies $\neg$(b).*

Assuming that X is not Kurtz random, there exists a computable sequence of finite sets of strings (D_i) (as a strong array) such that $\mu(D_i) \leq 2^{-i}$ and $X \in D_i$ for each i. Moreover, without loss of generality we

may assume that for each i, the strings in D_i have the same length d_i. For each i, the number of strings in D_i is at most $2^{d_i - i}$. Without loss of generality we may assume that for each i we have

- $d_i < d_{i+1}$ and $|D_i| = 2^{d_i - i}$ and $d_0 > 1$;

- every string in D_{i+1} has a prefix in D_i.

It suffices to construct a total Turing functional Φ such that for each $X \in \cap_i D_i$ there exists some Y such that $X = \Phi^Y$. As a basis for this functional, we define a partial computable tree T as a partial computable function from strings to strings, and an increasing sequence (q_s) of indices of the array (D_i), such that

(a) for each s and each $\rho \in D_{q_s}$ we have $T(\rho) \downarrow$ and $|T(\rho)| = |\rho| - s$;

(b) if $T(\rho) \downarrow$ then $\rho \in D_{q_s}$ for some s.

Let λ be the empty string. We define the sequence (q_s) along with T and an auxiliary increasing computable sequence (p_s) by simultaneous recursion, and use p_s as the length of $T(\rho)$ for each $\rho \in D_{q_s}$. In this construction we view the image $T(\rho)$ of ρ as a code for ρ.

At stage 0, let $q_0 = p_0 = 0$ and let $T(\lambda) = \lambda$. At stage $s + 1$ assume inductively that $q_j, p_j, j \leq s$ have been defined. Then let $q_{s+1} = p_s + s + 1$ and $p_{s+1} = d_{q_{s+1}} - s - 1$. Next, for each string $\rho \in D_{q_s}$ map the extensions of ρ in $D_{q_{s+1}}$ in lexicographical order onto the extensions of $T(\rho)$ of length p_{s+1}. Formally, let let $\eta_t, t < k$ be a lexicographical list of all the strings in $D_{q_{s+1}}$ which extend ρ. Moreover let $\theta_t, t < k$ be a lexicographical list of the first k extensions of $T(\rho)$ of length p_{s+1} and define $T(\eta_t) = \theta_t$ for each $t < k$. This completes stage $s + 1$ and the inductive definition of T.

First, let us verify that T is well defined, i.e. that at step $s + 1$ the required assignment of strings is possible. By our assumption about (D_i), given $\rho \in D_{q_s}$ there are at most $2^{d_{q_{s+1}} - q_{s+1}}$ many extensions of ρ in $D_{q_{s+1}}$. But since $p_{s+1} = d_{q_{s+1}} - s - 1$ and $q_{s+1} \geq 1$ we have $p_{s+1} \geq p_s + d_{q_{s+1}} - q_{s+1}$. This means that in step $s + 1$ there are enough extensions of $T(\rho)$ of the required length in order to perform the assignment $T(\eta_t) = \theta_t$ for each $t < k$.[10]

Moreover the tree T is clearly a computable map from strings of length $d_{q_{s+1}}$ to strings of length $p_{s+1} = d_{q_{s+1}} - s - 1$. So the properties (a), (b) above hold. Since step $s + 1$ always maps distinct extensions of ρ of a certain length to distinct extensions of $T(\rho)$ of a certain length, the map T is a tree in the sense that for η, θ such that $T(\eta) \downarrow, T(\theta) \downarrow$ we have $\eta \subseteq \theta$ if and only if $T(\eta) \subseteq T(\theta)$. Since T is a tree and the mapping in step $s + 1$ was done in lexicographical order, we have that for each s and each real $X \in \cap_{j \leq q_{s+1}} D_j$ the code $T(X \upharpoonright_{d_{q_s}})$ is defined and

for all s the string $X \upharpoonright_{d_{q_s}}$ is uniformly computable from $T(X \upharpoonright_{d_{q_s}})$ which has length $d_{q_s} - s$.

For each real $X \in \cap_i D_i$ let $T(X)$ be the limit of all $T(X \upharpoonright_{d_{q_s}})$ as s goes to infinity. Consider the intervals $I_s = [d_{q_s}, d_{q_{s+1}})$. Moreover let Φ be the total Turing functional such that for each X, s and $n \in I_s$ the oracle-use for the first n bits of X is $d_{q_s} - s$ and

$$\Phi^X(n) = \begin{cases} T(X)(n), & \text{if } X \in \cap_{j \leq q_{s+1}} D_j; \\ 0, & \text{otherwise.} \end{cases}$$

[10] Note that since (d_j) is increasing and $d_0 > 1$, for each s we have $q_{s+1} = p_s + s + 1 = d_{q_s} - s + s + 1 > d_{q_s} > q_s$ so $q_{s+1} > q_s + 1$. Also $p_{s+1} = d_{q_{s+1}} - s - 1 > d_{q_s} + 1 - s - 1 = d_{q_s} - s = p_s$ so both sequences (q_s) and (p_s) are increasing.

Then we can conclude that each $X \in \cap_i D_i$ is truth-table reducible to $T(X)$ and for each s the oracle-use for the computation of the first d_{q_s} bits of X is $d_{q_s} - s$. This concludes the proof of $\neg$(b) given $\neg$(a).

▶ **Proof that $\neg$(b) implies $\neg$(a).**

Assume that there exists some Y such that $X \leq_{tt} Y$ and for each c there exists n such that the oracle-use in this reduction for the computation of n-bit strings is bounded above by $n - c$. Let Φ denote a total Turing functional corresponding to the given truth-table reduction $X \leq_{tt} Y$. Also, let (n_i) be a computable increasing sequence of numbers such that for each i the oracle-use with respect to Φ for the computation of n_i-bit strings is bounded above by $n_i - i$. Then we may assume that the oracle-use in computations with respect to Φ is a non-decreasing computable function f such that $f(n_i) \leq n_i - i$ for each i, which is the same for all oracles. The functional Φ can be seen as a computable set of tuples $\langle \sigma, \tau \rangle$ indicating the fact that any real Y with prefix σ is Φ-mapped to an extension of τ. Moreover we can assume that if $\langle \sigma, \tau \rangle \in \Phi$ then for each $\rho \subseteq \tau$ there exists $\eta \subseteq \sigma$ such that $\langle \eta, \rho \rangle \in \Phi$. Using the fact that f is the oracle use in Φ, we may also assume that if $\langle \sigma, \tau \rangle \in \Phi$ then $|\sigma| = f(\tau)$.

For each i consider the set D_i of strings τ of length n_i such that $\langle \sigma, \tau \rangle \in \Phi$ for some σ. Then (D_i) is uniformly computable, since Φ is a total Turing functional. By the choice of (n_i) we have $|\sigma| + i \leq |\tau|$ for each such pair $\langle \sigma, \tau \rangle$. For each i let M_i be the set of strings σ such that $\langle \sigma, \tau \rangle \in \Phi$ for some $\tau \in D_i$. According to our hypothesis about Φ, all strings in M_i have the same length $f(n_i)$, so M_i is prefix-free. So

$$\mu(D_i) = \sum_{\tau \in D_i} 2^{-|\tau|} \leq \sum_{\sigma \in M_i} 2^{-|\sigma|-i} = 2^{-i} \cdot \sum_{\sigma \in M_i} 2^{-|\sigma|} \leq 2^{-i}.$$

Since the sets D_i are also finite and uniformly computable, (D_i) is a Kurtz test. By our hypothesis we have $\Phi^Y = X$ for some Y. Hence $X \in \cap_i D_i$ and this concludes the proof that X is not Kurtz random.

2.3 Proof of Theorems 1.6 and 1.7

The proof of Theorem 1.6 is a direct adaptation of the classic argument that shows the equivalence of the expressions of Martin-Löf randomness in terms of tests, martingales and prefix-free complexity. We need to show that the following are equivalent for a real X:

(a) $X \in \cap_i V_i$ for a computably granular Martin-Löf test (V_i);

(b) $M(X \restriction_{g(n)}) \geq n$ for all n, a c.e. (super)martingale M and a computable function g;

(c) $K(X \restriction_{g(i)}) \leq g(i) - i$ for a computable function g.

Suppose that (a) holds. Then without loss of generality we may assume that there exists a computable nondecreasing function g such that for each i, all strings in V_i have length $g(i)$. Then we can define $M(\sigma) = \sum_i \mu_\sigma(V_i)$. It can be easily verified that M is a c.e. martingale which meets (b) for any $X \in \cap_i V_i$. Conversely, assume that (b) holds and define U_i to be the set of strings of length $g(i)$ such that $M(X \restriction_{g(i)}) \geq i$. Also let $V_i = U_{2^i}$ for each i. Then $\mu(V_i) \leq 2^{-i}$ and (V_i) is granular with respect to the function $i \mapsto g(2^i)$. Now assume that (c) holds. Then define V_i be the set of strings σ of length $g(i)$ such that $K(\sigma) \leq |\sigma| - i$. Then $\mu(V_i) \leq 2^{-i}$ for each i and any X such that $K(X \restriction_{g(i)}) \leq g(i) - i$ is in V_i. Hence we can deduce (a). Conversely, suppose that $X \in \cap_i V_i$, $\mu(V_i) \leq 2^{-i}$ and all strings in V_i have length $g(i)$.

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Then using the Kraft-Chaitin theorem we can build a prefix-free machine M such that $K(\sigma) \leq |\sigma| - i$ for all i and for all $\sigma \in V_i$. This means that (a) implies (c).

Next, we turn to the proof of Theorem 1.7. We need to show that the following are equivalent:

(a) X is granularly random;

(b) for every Y such that $X \leq_{wtt} Y$ the Y-use in any such computation of $X \upharpoonright_n$ is bounded below by $n - c$ for some constant c and all n.

This proof is very similar to the argument we discussed in Section 2.2.

▶ *Proof that ¬(b) implies ¬(a).*

Assume that (b) does not hold. Then $\Phi(Y) = X$ for some Y and a Turing functional Φ with computable non-decreasing oracle-use f on all oracles such that for each c there exists some n such that $f(n) < n-c$. Let (n_i) be a computable increasing sequence such that $f(n_i) < n_i - i$ for each i. The functional Φ can be seen as a c.e. set of tuples $\langle \sigma, \tau \rangle$ indicating that for every oracle Y such that $\Phi(Y)$ is total, if Y is prefixed by σ then $\Phi(Y)$ is prefixed by τ. We may also assume that for each $\langle \sigma, \tau \rangle$ in Φ we have $f(|\tau|) = |\sigma|$. Hence for every i, every τ of length n_i and every σ such that $\langle \sigma, \tau \rangle \in \Phi$ the length of σ is at most $n_i - i$. Let V_i be the set of all τ of length n_i such that $\langle \sigma, \tau \rangle \in \Phi$ for some σ. Then by the choice of n_i, there can be at most $2^{n_i - i}$ many distinct strings in V_i. So $\mu(V_i) \leq 2^{-i}$ and (V_i) is a computably bounded Martin-Löf test. Moreover $X \in \cap_i V_i$ so X is not computably bounded random.

▶ *Proof that ¬(a) implies ¬(b).*

Conversely, assume that $X \in \cap_i V_i$ where (V_i) is a Martin-Löf test such that for each i the set V_i is finite. Without loss of generality we may assume that for each i, the strings in V_i have the same length v_i. Also, without loss of generality we may assume that for each i we have

- $v_i < v_{i+1}$ and $|V_i| \leq 2^{v_i - i}$ and $v_0 > 1$;

- every string in V_{i+1} has a prefix in V_i.

It suffices to construct a Turing functional Φ with computable uniform oracle-use, such that for each $X \in \cap_i V_i$ there exists some Y such that $X = \Phi^Y$.

Consider the tree T constructed in Section 2.2 in terms of the Kurtz test (D_i), and replace (D_i) with (V_i) and (d_i) with (v_i) in this construction. The only difference in the properties of T is that T is now merely a c.e. tree and not necessarily computable. Similarly to the construction of T in Section 2.2, consider the intervals $I_s = [v_{q_s}, v_{q_{s+1}})$. Define Φ be the Turing functional such that for each X, s and $n \in I_s$ the oracle-use for the first n bits of X is $v_{q_s} - s$ and

$$\Phi^X(n) = \begin{cases} T(X)(n), & \text{if } X \in \cap_{j \leq q_{s+1}} V_j; \\ \uparrow & \text{otherwise.} \end{cases}$$

Then we can conclude that each $X \in \cap_i V_i$ is truth-table reducible to $T(X)$ and for each s the oracle-use for the computation of the first v_{q_s} bits of X is $v_{q_s} - s$. This concludes the proof of ¬(b) given ¬(a).

3 Computing and compressing c.e. sets and c.e. reals

3.1 Proof of Theorem 1.3

We wish to prove a slightly stronger version of Theorem 1.3. We show that given a non-decreasing computable function f, the following are equivalent:

(a) $\sum_n 2^{-f(n)}$ is finite;

(b) every c.e. set X is computable by Ω with oracle-use bounded above by f;

(c) every c.e. set X is computable by some set Y with oracle-use bounded above by f.

The implication (a)→(b) is from [BFLP16] and (b)→(c) is trivial. It remains to show ¬(a)→ ¬(c). So suppose that the sum in (a) is infinite and let (Φ_e) be an effective enumeration of all Turing functionals with uniform oracle-use f. It suffices to construct a c.e. set X such that $X \neq \Phi_e^Y$ for all reals Y. Let (m_e) be a computable increasing sequence such that

$$\sum_{i\in[m_e,m_{e+1})} 2^{-f(i)} > 2 \qquad \text{for all } e.$$

The enumeration (X_s) of X is based on the effective enumeration of (Φ_e). At stage $s + 1$, consider the least $e \leq s$ such that $\Phi_e^\tau[s]$ is a suffix of $X_s \restriction_t$ for some $t \in I_e - X_s$ and some τ of length $f(t)$. If such e exists, enumerate the least such t into X.

Clearly X is a c.e. set. For a contradiction, suppose that $\Phi_e^Y = X$ for some Y and some e. This means that every number in I_e will be enumerated into X at some stage. Let $s_i, i < |I_e|$ be an increasing enumeration of the stages where these enumerations occurred. Moreover let $t_i, i < |I_e|$ be the enumerations that occurred in stages $s_i, i < |I_e|$. By standard assumptions about the functionals (Φ_j) and the construction, it follows that the sequence (t_i) is increasing. For each $i < |I_e|$, at stage s_i some number $t_i \in I_e$ was enumerated into $X_{s_i} - X_{s_i-1}$ so there exists a string τ_i of length $f(t_i)$ such that $\Phi_e^{\tau_i}[s_i - 1]$ is a suffix of $X_{s_i-1} \restriction_{t_i}$ but not a suffix of $X_s \restriction_{t_i}$ for any $s \geq s_i$. Since (t_i) is increasing and f is non-decreasing, it follows that the sequence $(|\tau_i|)$ is non-decreasing and $\tau_i, i < |I_e|$ is a prefix-free set of strings. Hence by Kraft's inequality we have

$$\sum_{i<|I_e|} 2^{-|\tau_i|} \leq 1.$$

On the other hand by the choice of I_e we have

$$\sum_{i<|I_e|} 2^{-|\tau_i|} = \sum_{j\in I_e} 2^{-f(j)} > 2$$

which is a contradiction. Hence we may conclude that there does not exist any real Y such that $\Phi_e^Y = X$ which completes the proof of ¬(a)→ ¬(c).

3.2 Proof of Theorem 1.8

Recall the statement of Theorem 1.8:

Every left-c.e. real X can be computed from Ω with oracle-use $g(n) = \max_{i\leq n} K(X \restriction_n)$.

Let g, Ω, X be as above and let $(X_s), (\Omega_s)$ be computable nondecreasing dyadic rational approximations that converge to X, Ω respectively. We construct a Solovay J test as follows.

At stage $s + 1$, consider the set I_s of $n \leq s$ such that one of the following holds:

  (a) $X_{s+1} \upharpoonright_n = X_s \upharpoonright_n$ and $K_{s+1}(X_{s+1} \upharpoonright_n) < K_s(X_s \upharpoonright_n)$;

  (b) $X_{s+1} \upharpoonright_n \neq X_s \upharpoonright_n$ and $K_{s+1}(X_{s+1} \upharpoonright_n) < \infty$.

For each $n \in I_n$ enumerate $\Omega_{s+1} \upharpoonright_n$ into J.

First we verify that J is a Solovay test. According to the construction, every string σ enumerated in J at stage $s + 1$ corresponds to clause (a) or clause (b). In either case, any such σ has the length of a corresponding currently shortest description of $X_{s+1} \upharpoonright_n$ for some n. Hence any string $\sigma \in J$ is uniquely associated with a string of the same length, in the domain of the underlying universal machine U. Since the weight of U is bounded by 1, the same will be true of the weight of J. Since J is also a c.e. set, it is a Solovay test.

It remains to show that for each n the string $\Omega \upharpoonright_{g(n)}$ can uniformly compute $X \upharpoonright_n$. Since Ω is Martin-Löf random there must be a stage t_0 in the construction after which no initial segment of Ω is ever enumerated in J after t_0. Given n and any initial segment τ of Ω of length at least $g(n)$, consider the first stage $s_0 > t_0$ such that τ is a prefix of Ω_{s_0}. Since Ω is a left-c.e. real we have that τ will be a prefix of Ω_{s_0} for all $s \geq s_0$. We claim that $X \upharpoonright_n = X_{s_0} \upharpoonright_n$.

For a contradiction, suppose that this is not the case. Then there exists a least stage $s_1 \geq s_0$ such that $X_{s_1+1} \upharpoonright_n \neq X_{s_1} \upharpoonright_n$. If $K_{s_1+1}(X_{s_1+1} \upharpoonright_n) = K(X \upharpoonright_n)$ then the construction will enumerate an initial segment of Ω of most $|\tau|$ into J at stage $s_1 + 1$. This would contradict our assumption about the choice of s_0. If on the other hand $K_{s_1+1}(X_{s_1+1} \upharpoonright_n) \neq K(X \upharpoonright_n)$ then consider the least stage $s_2 > s_1$ such that $K_{s_2+1}(X_{s_2+1} \upharpoonright_n) = K(X \upharpoonright_n)$. At stage $s_2 + 1$ the construction will enumerate the first $K(X \upharpoonright_n) \leq g(n)$ bits of Ω into J. This also contradicts our choice of stage s_0. It follows that $X \upharpoonright_n = X_{s_0} \upharpoonright_n$, which concludes the proof that X is computable from Ω with oracle-use g.

3.3 Proof of Theorem 1.9

Given a computable non-decreasing function f we need to show that the following are equivalent:

  (a) the prefix-free complexity of every c.e. set is bounded by $f + \mathbf{O}(1)$;

  (b) $\sum_n n \cdot 2^{-f(n)}$ is finite.

Note that (b) implies that $f(n)$ is an upper bound for $K(n)$. On the other hand, in order to describe the first n bits of a c.e. set we merely need the shortest prefix-free description of n concatenated by a code of the number of elements in $A \upharpoonright_n$. Since the latter has length at most $\log n$, we see that (b) implies (a).

Next, we show that ¬(b) implies ¬(a). Assuming that the sum in (b) is infinite we wish to construct a c.e. set A whose initial segment prefix-free complexity is not $f + \mathbf{O}(1)$. In the construction of A we will use the following fact.

Lemma 3.1. *If $\sum_n n \cdot 2^{-f(n)}$ is infinite then for every $c \in \mathbb{N}$ the sum $\sum_{n>c}(n - c) \cdot 2^{-f(n)}$ is infinite.*

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Proof. For a contradiction suppose that this does not hold for some c. Then

$$\sum_n n \cdot 2^{-f(n)} - \sum_{n>c}(n-c) \cdot 2^{-f(n)} = \sum_{n\leq c} n \cdot 2^{-f(n)} + \sum_{n>c} c \cdot 2^{-f(n)}$$

is infinite, which means that

$$\sum_{n>c} 2^{-f(n)} = \infty \qquad \text{and} \qquad \sum_{n>c}(n-c) \cdot 2^{-f(n)} = \infty$$

which contradicts our assumption. $\qquad\square$

We need to construct a c.e. set A which satisfies the following requirements:

$$\mathcal{R}_e : \quad \exists n \quad K(A \restriction_n) > f(n) + e.$$

By Lemma 3.1 there exists a computable increasing sequence (n_i) such that

$$\sum_{s\in(n_e,n_{e+1}]} (s - n_e) \cdot 2^{-f(s)} > 2^e \qquad \text{for all } e. \tag{3.3.1}$$

Let us say that $\mathcal{R}_e$ *requires attention* at stage $s + 1$ if $K(A \restriction_n)[s + 1] \leq f(n) + e$ for all $n \leq n_{e+1}$ and $[n_e, n_{e+1}) - A_s \neq \emptyset$. Note that the numbers we are going to enumerate into A for the satisfaction of $\mathcal{R}_e$ are from the interval $[n_e, n_{e+1})$ while in (3.3.1) we use interval $(n_e, n_{e+1}]$. This is because for each e, a change in the approximation to $A \restriction_{n_{e+1}}$ can only be achieved with the enumeration into A of one of the numbers in $[0, n_{e+1})$.

We can now define a computable enumeration (A_s) of A. At stage $s + 1$ consider the least $e \leq s$ such that $\mathcal{R}_e$ requires attention. If such e exists, enumerate the least number of $[n_e, n_{e+1}) - A_s$ into A. This completes the construction of A.

We verify that A meets requirement $\mathcal{R}_e$ for all e. Since for each e the interval $[n_e, n_{e+1})$ contains finitely many numbers, by definition each requirement can only require attention at finitely many stages. This is because when e is the least number such that $\mathcal{R}_e$ requires attention at stage $s + 1$, then a number from $[n_e, n_{e+1}) - A_s$ enters A. If $\mathcal{R}_e$ was not satisfied for some e, then it follows that $[n_e, n_{e+1}) \subseteq A$. But at each stage $s + 1$ where a number $n \in [n_e, n_{e+1})$ enters A we have $K(A \restriction_t)[s] \leq f(t) + e$ for all $t \in (n_e, n_{e+1}]$. So if such an enumeration of $n \in [n_e, n_{e+1})$ occurs at some stage stage $s + 1$, we can count an additional description of $A_s \restriction_t$ of length at most $f(t) + e$ for each $t \in (n, n_{e+1}]$ in the domain of in the underlying universal prefix-free machine. Indeed, this is because if $n \in [n_e, n_{e+1})$ was enumerated into A at some stage $s + 1$, the previous such enumeration was of the number $n + 1$ at some stage $t + 1 < s + 1$ and for each $m \in (n + 2, n_{e+1}]$ we have $A_s \restriction_m \neq A_t \restriction_m$. Of course if $n \in [n_e, n_{e+1})$ was the first number in this interval to be enumerated into A, then $n = n_{e+1} - 1$ and we count the description $A_s \restriction_{n_{e+1}-1}$ of length at most $f(n_{e+1}) + e$ for the first time. This means that by the stage where all of the numbers in $[n_e, n_{e+1})$ are enumerated into A, we can count descriptions in the domain of the universal machine of weight at least

$$\sum_{s\in(n_e,n_{e+1}]} (s - n_e) \cdot 2^{-f(s)-e}.$$

This contradicts (3.3.1) and the fact that the weight of the domain of a prefix-free machine is bounded above by 1. Hence we can conclude that the c.e. set A that we enumerated satisfies $\mathcal{R}_e$ for each e.

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3.4 Proof of Theorem 1.10

We need to show that:

    (a) if **a** is array non-computable then it contains a c.e. set whose prefix-free complexity is not bounded above by $2 \log n + \log \log n$;

    (b) if **a** is array computable then the prefix-free complexity of all of its c.e. members is bounded above by $\log n + 2 \log \log n + \mathbf{O}(1)$.

For (a), note that $\sum_i 2^{-\log i - \log \log i} = \infty$, so we only need to adapt the construction for the proof of Theorem 1.9 in Section 3.3, inside any c.e. array non-computable c.e. degree. In the following we fix $f(n) = \log n$ in Theorem 1.9 and the construction of Section 3.3. Note that it suffices to only satisfy infinitely many requirements $\mathcal{R}_e$ from Section 3.3. Recall from [DJS96] that if (I_n) is a computable sequence of intervals such that $I_n < I_m$ whenever $n < m$ then

> every c.e. array non-computable degree **a** contains a c.e. set A such that for
> each c.e. set W we have $A \cap I_n = W \cap I_n$ for infinitely many n.   (3.4.1)

Without loss of generality we may assume that $n_{e+1} - n_e > n_e - n_{e-1}$ for all $e > 0$ in the construction of Section 3.3. Then we can set $I_e = [n_e, n_{e+1})$ and apply (3.4.1) in order to fix a c.e. set A in **a** with the stated property. Next, we construct a c.e. set W in stages as follows, mimicking the construction of Section 3.3. At stage $s + 1$, let $e \leq s$ be the least number such that $K(A \restriction_n)[s + 1] \leq f(n) + e$ for all $n \leq n_{e+1}$ and $I_e \nsubseteq A_s$, and if such e exists put the smallest element of $I_e - A_s$ into W.

By (3.4.1) we have $A \cap I_e = W \cap I_e$ for infinitely many n. Given e such that $A \cap I_e = W \cap I_e$, the assumption that $\mathcal{R}_e$ is not satisfied leads to a contradiction by the same argument that we used in Section 3.3. Hence we can conclude that there are infinitely many e such that A satisfies $\mathcal{R}_e$, which means that the prefix-free complexity of A is not bounded above by $\log n + f(n) + \mathbf{O}(1)$ which is $2 \log n + \mathbf{O}(1)$ by our choice of f.

For (b), following Kummer's argument from [Kum96] and replacing plain for prefix-free complexity, we get that if A is a c.e. set of array computable degree, $f(n)$ is a computable upper bound of $K(n)$ and g is a computable unbounded non-decreasing function, then $K(A \restriction_n)$ is bounded above by $f(n) + g(n) + \mathbf{O}(1)$. Note that $\sum_i 2^{-\log i - 1.5 \cdot \log \log i} < \infty$ so if we take $g(n) = 0.5 \cdot \log \log n$, the above fact shows that the prefix-free complexity of A is bounded above by $\log n + 2 \log \log n$.

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