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SOUND, TOTALLY SOUND, AND UNSOUND RECURSIVE EQUIVALENCE TYPES

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An independent subset I of V_∞ is called *sound* if I is contained in a r.e. independent set. If not, we shall call I *unsound*. We call an RET **A** *totally sound* if every independent set in **A** is sound. Clearly every RET containing an r.e. set is totally sound, and it had been suggested that the converse held: viz, every totally sound RET contained a recursive set.

However, we show that there are $2^{\aleph_0}$ sets $\{A_\pi\}$ such that if $B \leqslant_m A_\pi$ then RET(B) is totally sound. In the co-r.e. case it is shown that if A is co-r.e. nonrecursive and nonisolic, then RET(A) is not totally sound. Indeed, RET(A) contains an independent set which is a basis of a subspace (of codimension 1 in V_∞), no basis of which is sound. This result is deduced from a construction of a new type of r.e. subspace of V_∞. That is, we show that if R is a fully co-r.e. nonrecursive subspace of V_∞ there exists an r.e. subspace V such that $V \oplus R = V_\infty$ and V has the property that if W is an r.e. subspace with $W \supset V$, then $\dim(V_\infty/W) < \infty$ implies $W = V_\infty$.

On the other hand, the co-simple isols provide a nice dichotomy. First, it is shown that (in each high r.e. degree) there is a co-simple totally sound RET. Second it is shown that if $\boldsymbol{a}$ is any nonzero r.e. degree, there exists a nonzero r.e. degree $\boldsymbol{b} \leqslant_T \boldsymbol{a}$ such that $\boldsymbol{b}$ bounds no nontrivial totally sound co-r.e. RET. That is, if B is r.e. with Turing degree $\boldsymbol{b}$ and C is co-r.e. with $C \leqslant_T B$, then either (i) C is recursive or (ii) RET(C) is not totally sound.

Finally, we extend our results to general algebraic settings; first to a class of Steinitz closure systems, and later to a general model-theoretic setting (without dependence).

1. Introduction

Two sets A and B are called recursively equivalent (written $A \cong B$), if there is a partial recursive 1–1 function f with $\operatorname{dom} f \supset A$, $\operatorname{ra} f \supset B$ and $f(A) = B$. The recursive equivalence type (RET) of A is the set $\{B \mid B \cong A\}$. RET's were introduced by Dekker [18] and since then have been extensively studied (for example, see Crossley [6] and [7]). A new direction in this study was initiated by Crossley [5], where Crossley analysed notions of recursive equivalence in algebraic *structures* rather than sets. In [5], he analysed recursive equivalence types of linear orderings, and demanded that the function in the definition above now be an order-isomorphism. There are several possible definitions for such RET's (called *constructive order types*), but we shall not discuss them here; rather we refer the reader to [5], [21] and [32]. Other structures were analysed, but each

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of these had the property that the algebraic closure of A was A. A general setting for this approach was given in [8].

Hassett [25] and Dekker [19] introduced techniques for dealing with structures where the algebraic closure of a set A is not necessarily A. In particular, Dekker [19] studied recursive equivalence types of vector subspaces of a fully effective vector space V_∞. By 'fully effective' we mean an infinite-dimensional vector space recursively isomorphic to $\bigoplus_\omega F$, where F is a (fixed) computable field. (There are other possible approaches to such RET's; see (for example) Remmel [32].) Here the additional vector space structure gave rise to two very different types of subspaces: ones with a basis *contained* in an r.e. independent set, and ones without such a basic. Following Crossley and Nerode [9], we shall call the first type *sound* and the latter *unsound*. (Dekker calls the former *α-spaces*.)

One idea of Dekker was to identify 'dimension' with the RET of a basis. Dekker found that to obtain many of his results he needed to concentrate on sound subspaces. Indeed, in [19], he concentrated on *isolic* sound subspaces (that is, ones with no infinite dimensional r.e. subspace; Crossley and Nerode call these *Dedekind*). He showed for such spaces, that any two sound bases were recursively equivalent. Later Hamilton [22, 23] showed if A and B are sound bases of a subspace V, then $A \cong B$. Since then, most work (for example [9], [24] and [34]) has concentrated almost completely on sound subspaces.

This paper addresses itself to a fundamental question concerning unsound subspaces:

What RET's contain bases of unsound subspaces?

This is a question of Nerode and of Dekker, which probably first appeared explicitly in Hamilton [22]. Evidently no RET containing an r.e. set can be unsound (that is contain an unsound basis). There is a very easy way to construct a plethora of unsound independent sets given in Downey [11]: let C be an immune co-r.e. subset of a recursive basis. Then there is an r.e. subspace V with $V \oplus (C)^* = V_\infty$ which has the property that if E is any infinite proper subset of C, no basis $V \oplus (E)^*$ is sound (where, for $X \subset V_\infty$, $(X)^*$ denotes the subspace generated by X). Also see Theorem 3.3 of the present paper.

This, and various other, constructions suggest that there are virtually no nontrivial restrictions on the RET's of bases of unsound subspaces, especially if we insist that the RET's be nonrecursive and nonimmune. However, this is not the case.

Definition. An RET **A** is called *totally sound* if every independent set in **A** is sound.

We show that there exist $2^{\aleph_0}$ totally sound RET's of both the isolated and nonisolated variety (Theorem 2.1). Indeed, we show that there are $2^{\aleph_0}$ sets $\{A_\pi\}$

such that if B is any set with B many-one reducible to A, then the RET of B is totally sound, and these sets may be chosen not to have minimal m-degree (Corollary 2.2).

These results lead us to wonder what happens in the case where we ask our RET's to be co-r.e. In the case where $\mathbf{A}$ is a co-r.e. nonisolic RET, we show that $\mathbf{A}$ is not totally sound, that is, some unsound basis occurs in $\mathbf{A}$ (Corollary 2.4). In fact, each such $\mathbf{A}$ contains a basis of a co-r.e. subspace every basis of which is unsound (Corollary 2.7). The co-simple case is quite different.

Remmel [30] shows, on the one hand, that every r.e. degree contains a co-r.e. subspace each basis of which is unsound. (See the remarks following the proof of Theorem 2.3.) On the other hand we prove two classification results. We show that in every high r.e. degree there is a co-simple totally sound RET (Theorem 2.8). We cannot classify the r.e. degrees containing such totally sound co-r.e. RET's, but we can show that this class does not include all r.e. degrees. The second result (Theorem 2.9) is: if δ is any r.e. nonzero degree, then there is a nonzero r.e. degree $\delta' \leqslant_{\mathrm{T}} \delta$ bounding no nontrivial totally sound co-r.e. RET, that is, if B is co-r.e. of degree $\leqslant_{\mathrm{T}} \delta'$, then if RET($B$) is totally sound, B is recursive.

In Section 3, we generalize our results, first to a class of matroids, and then to a very wide class of recursive models. This involves some generalizations of the ideas of soundness and bases, and we do this in much the same way as for the recursive 'universal algebra' of Downey and Remmel [17].

A few bits of notation. W_e will denote the eth r.e. set, and $\{\phi_e\}_{e\in\omega}$ an enumeration of the unary partial recursive functions. We will denote by I_e the eth independent r.e. subset of V_∞. If $A \subset V_\infty$, $(A)^*$ denotes the subspace generated by A. 'dim' means dimension. The reader unfamiliar with V_∞ can think of it as the weak direct sum of ω copies of Q (if the field is infinite) or ω copies of the 2-element field GF(2) (if the field is finite).

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2.

Our first task is to give the basic construction.

Theorem 2.1. *There exist $2^{\aleph_0}$ totally sound RET's. Moreover, these may be chosen as either isolated or non-isolated.*

Proof. We build a collection of sets $\{A_\pi\}$ with $A_\pi = \bigcup_s A_\pi^s$ for each path π through the complete binary tree 2^ω. We ensure that if $\pi_1 \neq \pi_2$, then $A_{\pi_1} \neq A_{\pi_2}$. This therefore gives $2^{\aleph_0}$ sets and so $2^{\aleph_0}$ RET's, since each RET has only $\aleph_0$ sets.

We meet the requirements:

$I_{\pi,e}$: card$(A_\pi) \geq e$.
$P_{\pi,e}$: $W_e \not\subseteq A_\pi$ if card$(W_e) = \infty$.
$N_{\pi,e}$: One of the following fails to hold;
(i) ϕ_e is 1–1,
(ii) dom $\phi_e \supset A_\pi$,
(iii) $\phi_e(A_\pi)$ is independent,
(iv) $\phi_e(A_\pi)$ is unsound.

Let $p(0), \ldots, p(2^e)$ denote the strings of length e with lexicographic order. At stage e we will have constructed $A^e_{p(j)}$ for $j \leq 2^e$, so that if π extends $p(j)$, then $A_\pi = A_\xi$ implies ξ extends $p(j)$. For each π we construct an auxilliary set $K_\pi = \bigcup_s K^s_\pi$.

Construction, stage $e+1$

Step 1. *For* $P_{\pi,e}$, if W_e infinite, find the least $x_e \in W_e$ with $x_e \notin \bigcup_{j \leq 2^e}(A^e_{p(j)} \cup K^e_{p(j)})$. Now for each $j \leq 2^e$ set $T_{p(j)} = K^e_{p(j)} \cup \{x_e\}$. If W_e is not infinite, set $T_{p(j)} = K^e_{p(j)}$.

Step 2. Set $Q = \omega - \bigcup_{j \leq 2^e}(A^e_{p(j)} \cup T_{p(j)})$. There are four cases.

Case (i). ϕ_e is not 1–1. Set $M_{p(j)} = A^e_{p(j)}$ for all $j \leq 2^e$.

Case (ii). ϕ_e is 1–1, but there exists $y \in Q$ with $\phi_e(y)\uparrow$. In this case set $M_{p(j)} = A^e_{p(j)} \cup \{y\}$ for all $j \leq 2^e$.

Case (iii). ϕ_e is 1–1 and for all $y \in Q$, $\phi_e(y)\downarrow$, but there exist $j \leq 2^e$ and $q_1(j), \ldots, q_{n_j}(j) \in Q$ such that

$$\{\phi_e(q_1(j)), \ldots, \phi_e(q_{n_j}(j)), \phi_e(x) \mid x \in A^e_{p(j)}\}$$

is independent. In this case, for each such $j \leq 2^e$, our action is to set $M_{p(j)} = A^e_{p(j)} \cup \{q_1(j), \ldots, q_{n_j}(j)\}$.

Case (iv) Otherwise. For each $j \leq 2^e$ for which cases (i), (ii) and (iii) did not apply, set $M_{p(j)} = A^e_{p(j)}$.

Step 3. Let $\{z(0), \ldots, z(2^{e+1})\}$ list in order the least 2^{e+1} elements of $\omega - \bigcup_{j \leq 2^e}(M_{p(j)} \cup T_{p(j)})$. Define for each $j \leq 2^e$, and $i = 0, 1$,

$$K^{e+1}_{p(j)\hat{}i} = T_{p(j)},$$

$$A^{e+1}_{p(j)\hat{}0} = M_{p(j)} \cup \{z(2j)\} \quad \text{and} \quad A^{e+1}_{p(j)\hat{}1} = M_{p(j)} \cup \{z(2j+1)\}.$$

This concludes the construction. By Step 1 we ensure that all the $P_{\pi,e}$ are met. By Step 3 we meet all the $I_{\pi,e}$ and ensure that $A_{\pi_1} \neq A_{\pi_2}$ unless $\pi_1 = \pi_2$. The heart of the construction is Step 2. In cases (i), (ii) and (iii), we diagonalize (forever) ϕ_e from being a candidate to witness the unsoundness of any A_π with $p(j)$ an initial segment of π. Let j be such that cases (i), (ii) and (iii) did not apply. It follows that for all finite subsets Y of Q, $\phi_e(A^e_{p(j)} \cup Y)$ is independent, ϕ_e is 1–1 and $\forall y \in Q$, $\phi_e(y)\downarrow$. Therefore $\phi_e(A^e_{p(j)} \cup Q)$ is independent and, since $Q =^* \omega$, $\phi_e(A^e_{p(j)} \cup Q)$ is r.e. Therefore for any π extending $p(j)$, as $A_\pi \subset (A_\pi \cup$

Q), it follows that $\phi_e(A_\pi)$ is an independent subset of the r.e. independent set $\phi_e(A_\pi \cup Q) = \phi_e(A^e_{p(j)} \cup Q)$, and so $\phi_e(A_\pi)$ is sound.

This gives the case where A_π is isolated. It is easy to modify the construction to start with A_0 as an infinite, coinfinite recursive set (or, indeed any coinfinite set). □

This result may be sharpened somewhat to show:

Corollary 2.2. *There exist* $2^{\aleph_0}$ *totally sound m-degrees, in fact, there are* $2^{\aleph_0}$ *m-degrees* δ_π *such that if A has m-degree* $\leqslant \delta_\pi$, *then the RET of A is totally sound.*

Proof. For simplicity, we construct only one such set. We make A infinite and immune as before and satisfy

R_e: If ϕ_e is total, and γ_e is 1–1,
then $\gamma_e(\phi_e(A))$ is sound if it is defined and independent.

Modifications, stage $e+1$. Assume we have satisfied P_e as in Step 1 of Theorem 3.1, and thus defined A^e and T. We might as well also assume ϕ_e is total and γ_e is 1–1. Set $Q = \omega - (A^e \cup T)$. Thus

Case (ii)$'$. There exists $y \in Q$ with $\gamma_e(\phi_e(y))\uparrow$. In this case, set $M = A^e \cup \{y\}$.

Case (iii)$'$. ϕ_e is total, γ_e is 1–1 and, for all $y \in Q$, $\gamma_e(\phi_e(y))\downarrow$, but there exist $q_1, \ldots, q_n \in Q$ such that $\phi_e(q_1), \ldots, \phi_e(q_n)$ are all distinct, and

$$\{\gamma_e(\phi_e(q_1)), \ldots, \gamma_e(\phi_e(q_n))\} \cup \gamma_e(\phi_e(A_e))$$

is dependent. In this case, set $M = A^e \cup \{q_1, \ldots, q_n\}$.

Case (iv)$'$. Otherwise, set $M = A^e$.

Now perform Step 3 similarly as before. The key point again is that in case (iv)$'$, $\gamma_e(\phi_e(Q \cup A^e))$ is an r.e. independent set containing $\gamma_e(\phi_e(A)) = \gamma_e(\bigcup_s A^s))$. □

We remark that we could interweave other requirements with the above. For example we could ensure the m-degrees constructed are not minimal, or do not consist of only one 1-degree, or control the T-degrees above $\mathbf{0}''$, since the entire construction could be performed with a $\mathbf{0}''$-oracle, and admits coding. We leave these modifications to the reader. Our next goal is to analyse co-r.e. aspects of these results. The next observation is quite easy, but it demonstrates how the 'oracle' case differs from the co-r.e. one.

Theorem 2.3. *Let* $A = R \oplus C$ *where R is infinite recursive and C is co-r.e. and immune. Then A is recursively equivalent to a basis of a subspace each basis of which is unsound.*

Proof. Recall, from example [11], that an r.e. subspace V is called supermaximal if $\dim(V_\infty/V) = \infty$ and for all r.e. subspaces W of V_∞, if $W \supset V$, then either

$\dim(W/V) < \infty$, or $W = V_\infty$. Let B be a recursive basis of V_∞, and $A = R \oplus C$ as above. Without loss of generality, we may consider $A \subset B$. Let $x \in B - C$. Put $D = C \cup \{x\}$. Now D is immune and $(D)^*$ is fully co-r.e., namely generated by a co-r.e. subset of a recursive basis. By a result of Downey [11], there exists an r.e. supermaximal subspace V with $V \oplus (D)^* = V_\infty$. Let $Q = V \oplus (C)^*$. Suppose Q is sound. Then let E be a sound basis of Q and E' an r.e. independent set containing E. Then, clearly, as V is supermaximal, $(E')^* = V_\infty$. But $\dim(V_\infty/Q) = 1$, and so $(E)^*$ is r.e. As V is supermaximal again, $(E)^* = V_\infty$; a contradiction. Finally Dekker [19] has shown that V has a recursive basis K. Let $F = K \cup C$. Then $F \cong A$ and $(F)^* = Q$. □

Remarks. (i) The above techniques may be utilized to analyse other aspects of V_∞ and soundness. Notice that Q has the property that it is not r.e. and if V is r.e. and $V \supset Q$, then $V = V_\infty$. In [11] such subspaces were called *superunsound*. This, and a stronger property called *nowhere soundness*, were analysed in [11]. The real difference is that this approach considers 'structural' soundness, viz, soundness as a subspace, versus soundness of only independent sets (cf. Section 4). (See also Theorem 2.6.)

(ii) These techniques have since found applications to the lattice of r.e. subspaces (cf. [10, 11], [14], [15] and [28]).

(iii) For our purposes, one interesting application to RET's, is a very easy proof that the Karp–Myhill theorem fails even for (co-)r.e. subspaces of V_∞. (The fact that this fails for subspaces r.e. in $\mathbf{0}'$, was first established by Soare in [34]. He also proved this for 'Dedekind cuts'.) Recall from [20], that the theorem states that if $A \cong B$ and $\bar{A} \cong \bar{B}$, then $A \equiv_1 B$. Let C be a co-r.e. nonrecursive subset of a recursive basis. It is established in [11] that there exist r.e. subspaces V_1, V_2 with V_1 recursive, V_2 nonrecursive and $V_1 \oplus (C)^* = V_2 \oplus (C)^* = V_\infty$. Here $V_1 \not\equiv_T V_2$. This may be improved to $V_1 |_T V_2$ (cf. [15]).

(iv) As a final remark, this result gives a proof that there exist unsound co-r.e. bases of unsound co-r.e. subspaces in each nontrivial co-r.e. m-degree. By the well known results of Jockusch and Simpson (as C has infinite subsets in each T-degree $\geq$ T-deg(C)), the proof also produces an unsound subspace in each T-degree bounding an r.e. T-degree. This may be improved. Remmel [31] gave a proof that there exists, in each r.e. nonzero T-degree, an unsound co-r.e. (immune) subspace. His proof may be easily converted into an oracle one, to produce an unsound (immune) subspace in each T-degree. We refer the reader to [31].[1]

We may extend Theorem 2.3 to cover the case where no decomposition of A is available as follows.

[1] This may also be proved using the result (unpublished) of Downey and Remmel, that if V is any infinite-dimensional r.e. subspace, then V has a basis of any given weak truth table degree, applied to a recursive subspace of codimension 1.

Corollary 2.4. *Let* $\mathbf{A}$ *be any nonisolic nonrecursive co-r.e. RET. Then* $\mathbf{A}$ *contains an unsound co-r.e. independent set.*

Proof. Let R be a co-r.e. subset of a recursive basis B of V_∞ of RET $\mathbf{A}$. Let $V = (B - R)^*$. This is an infinite-dimensional r.e. subspace, and so by Dekker [19], it has a recursive basis Q. The key observation here is that Q is not fully extendible. That is, Q is contained in no r.e. basis of V_∞. For suppose it were contained in, say, B' an r.e. (and hence recursive) basis of V_∞. Then $(B' - Q)$ is also r.e., and from this it is easy to show that $(Q)^*$ is recursive since $(Q)^* \oplus (B' - Q)^* = V_\infty$: to see if $x \in (Q)^*$ find the unique λ_i with $x = \sum \lambda_i b_i$ and $b_i \in B'$. Then $x \in (Q)^*$ iff all the b_i are in Q.

Now, let $x \in R$, and set $T = Q \cup (R - \{x\})$. Now, we claim T is an unsound independent set. If not, then as above, for some $y \in V_\infty$, $T \cup \{y\}$ is a recursive basis of V_∞ containing Q, contradicting choice of Q. Therefore we conclude that T is unsound.

It therefore remains to show that $R \cong T$. As R is not immune, R has an infinite recursive subset F, where the above $x \in R$ is a member of F say. Let $Q = \{q_0 < q_1 < \cdots\}$ and $F = \{f_0 < f_1 < \cdots\}$ be recursive listings of Q and F in order of magnitude. Define a recursive 1–1 function h via $h(x) = x$, if $x \notin F \cup Q$; $j(f_{2i}) = q_i$ and $h(f_{2i+1}) = f_i$ for all $i \in \omega$. It is easy to see that h makes $R \cong T$. □

The reader should note that Corollary 2.4 is *not* the complete extension of 2.3 to arbitrary nonisolic co-r.e. RET's. This is because in 2.4 we should that $\mathbf{A}$ contained an unsound *independent set*; but in 2.3 we showed that $\mathbf{A}$ contained a basis of an *unsound subspace*, that is, a subspace *each basis of which* is unsound. In Corollary 2.7 we shall give the complete extension of 2.3 to unsound subspaces. However, this result is much more difficult to prove. Indeed, as we shall see, we shall need a new property of the lattice of r.e. subspaces to establish this. Why is this? First observe that in the proof of 2.4, $B - \{x\}$ is a sound basis of $(T)^*$, so the space used in 2.4 will not suffice. Nevertheless the *method* of 2.3 and 2.4 would work if we could perhaps select a 'good' r.e. complement of the co-r.e. space $(R)^*$. As a first try, we might attempt to use V such that no r.e. basis of V is contained in a recursive basis of V_∞. Unfortunately, as we shall show in the next example, there exists an r.e. V with no r.e. basis contained in a recursive basis of V_∞, but such that $V \oplus (R - \{x\})^*$ is sound for some $x \in R$:

Example. Let S be an r.e. nonrecursive nonsimple subset of a recursive basis B of V_∞. Using a result of Downey [12], $(S)^*$ may be decomposed into a pair of r.e. subspaces $V_1 \oplus V_2 = (S)^*$, where no basis of either V_1 or V_2 may be extended to an r.e. basis of V_∞. Now we use a technique from [11]:

Construct $B(V_1) = \bigcup_s B_s(V_1)$ in stages, as a subset of B. At each stage s, we denote by $\{b_{0,s} < b_{1,s} < \cdots\}$ the set $B - B_s(V_1)$. Let $V_1 = \bigcup_s V_{1,s}$ be a standard enumeration of V_1.

Stage 0. Set $B_0(V_1)=\emptyset$.

Stage $s+1$. For $e\leqslant s$ find the least e (if any), such that

(i) $b_{e,s}\in(V_{1,s}\cup\{b_{i,s}\mid i<e\})^*$, and

(ii) for $i<e$, $b_{i,s}\notin(V_{1,s}\cup\{b_{j,s}\mid j<i\})^*$.

If no such e exists, set $B_{s+1}(V_1)=B_s(V_1)$. If e exists, set $B_{s+1}(V_1)=B_s(V_1)\cup\{b_{e,s}\}$ and regenerate $\{b_{i,s+1}\}$. To complete the construction, set $B(V_1)=\bigcup_s B_s(V_1)$. Now, it is easy to show that $B-B(V_1)$ is co-r.e., that $B-B(V_1)\supset B-S$, and that $B-B(V)$ is a basis of V_∞ over V_1. Now let $x\in B-S$. Then $(B-\{x\})^*\supset V_1$, and $(B-\{x\})$ is constructed by a technique similar to Corollary 2.4, no r.e. basis of V_1 is fully extendible and yet (again) $(B-\{x\})^*$ is sound. □

To give the desired full extension of Theorem 2.3, we use a new property of the lattice of r.e. subspaces.

Theorem 2.5. *Let R be any nonrecursive co-r.e. subset of a recursive basis B. Then there exists an r.e. subspace V of V_∞ such that*

(i) $V\oplus(R)^*=V_\infty$.

(ii) *If W is any r.e. subspace of V_∞ with $W\supset V$ and $\dim(V_\infty/W)<\infty$, then $W=V_\infty$.*

Proof. We build $V=\bigcup_s V_s$ in stages. Let R and B be as above. We enumerate $R=\bigcap_s R_s$ in stages, given by the co-r.e. limit lemma; thus a recursive sequence of cofinite sets with $R_0=B$, $\mathrm{card}(R_{s+1}-R_s)=1$ and $R_{s+1}\subset R_s$. The unique $x\in R_{s+1}-R_s$ we denote by $z(s)$. We have an absolute commitment to ensure that $(R)^*\oplus V=V_\infty$. To do this, at each stage s, we define $\mathrm{supp}_s(y)$ to be the support of y relative to R_s over $(V_s)^*$. That is, the unique smallest $\{b_i^s\}$ such that $y=\sum\lambda_i b_i^s \bmod (V_s)^*$. We ask that y only enter $(V_{s+1})^*-(V_s)^*$ if $z(s)\in\mathrm{supp}_s(y)$. Exchange will ensure that if this plan is faithfully carried out, then V will be a complement of $(R)^*$. Together with this, we must meet the requirements:

$Q_{\langle e,x\rangle}$: If $x\in I_e$ and $(I_e)^*=V_\infty$, then $(I_e-\{x\})^*\not\supset V$.

Define a modification J_e of the I_e as follows:

Stage 0. Set $t(0)=0$ and set $J_{e,0}=\emptyset$.

Stage $s+1$. Assuming we have defined $t(s)$ and $J_{e,s}$, define $t(s+1)=t(s)$ and $J_{e,s+1}=J_{e,s}$ if $t(s)\notin(I_{e,s})^*$; and if $t(s)\in(I_{e,s})^*$, define $J_{e,s+1}=I_{e,s}$ and set $t(s+1)=t(s)+1$.

Notice that $\mathrm{card}(J_e)=\infty$ iff $(I_e)^*=V_\infty$ (for if $(I_e)^*\neq V_\infty$ we get stuck on some $t(s)$). Now define $\mathrm{supp}(e,x)$ to be the support of x relative to J_e (if this is defined). We shall meet[2]:

$Q_{\langle e,x\rangle}$: If $x\in J_e$ and $\mathrm{card}(J_e)=\infty$, then there exists $y\in V$ such that $x\in\mathrm{supp}(e,y)$.

[2] Although not directly pertaining to RET's, we remark that a similar apparatus can be used to analyse certain elementary properties of, for example, supermaximal subspaces. We refer the reader to the author's 'Bases of supermaximal subspaces I' (J. Symbolic Logic 49 (1984) 1146–1159) and 'II' (to appear Z. Math. Logik Grund. Math.).

At each stage s, we define supp(e, s, y) to be the (partial) support of y relative to $J_{e,s}$, which is thus defined only if $y \in (J_{e,s})^*$. We say $Q_{\langle e,x\rangle}$ is *satisfied* at stage s if $\exists y\,(y \in V_s \;\&\; x \in \text{supp}(e, s, y) \;\&\; x \in J_{e,s})$. We say $Q_{\langle e,x\rangle}$ *requires attention* if $\langle e, x\rangle$ is least such that $x \in J_{e,s}$ and supp$(e, s, z(s))$ is defined, with $Q_{\langle e,x\rangle}$ not satisfied at stage s.

Construction, stage $s+1$

If no $Q_{\langle e,x\rangle}$ for $\langle e, x\rangle \leqslant s$ requires attention, set $V_{s+1} = (V_s \cup \{z(s)\})^*$. If $Q_{\langle e,x\rangle}$ requires attention via $z(s)$, there are three cases:

Case (i). $z(s) \in \text{supp}_s(x)$. Set $V_{s+1} = (V_s \cup \{x\})^*$.

Case (ii). $z(s) \notin \text{supp}_s(x)$ and $x \notin \text{supp}(e, s, z(s))$. Set $V_{s+1} = (V_s \cup \{z(s) + x\})^*$.

Case (iii). $z(s) \notin \text{supp}_s(x)$ and $x \in \text{supp}(e, s, z(s))$. Set $V_{s+1} = (V_s \cup \{z(s)\})^*$.

To conclude the construction, set $V = \bigcup_s V_s$.

Obviously, we ensure that, in cases (i) and (iii), $z(s) \in \text{supp}_s(y)$ if $y \in V_{s+1} - V_s$. In case (ii), since $z(s) \notin \text{supp}_s(x)$, $z(s) \in \text{supp}_s(x + z(s))$. Thus $R \oplus V = V_\infty$. Also, observe that if $Q_{\langle e,x\rangle}$ requires attention, then it is met: in cases (i) and (iii) this is obvious, in case (ii) it again follows by exchange. Therefore it remains to prove that if card$(J_e) = \infty$, then for all $x \in J_e$, $Q_{\langle e,x\rangle}$ will require attention. By induction let $\langle e, x\rangle$ be least, for which $Q_{\langle e,x\rangle}$ fails, and go to a stage t where the requirements of higher priority cease to matter and $x \in J_e$. We need to show that $Q_{\langle e,x\rangle}$ will require attention some time. This means some stage $s > t$ must occur where supp$(e, s, z(s))$ is defined. We claim the failure of this to happen after stage t would mean R is recursive: to compute if $y \in R$, first check whether $y \in B$. If not, then $y \notin R$. If $y \in B$, compute the least stage $s \geqslant t+1$ such that supp(e, s, y) is defined. Then $y \in R \leftrightarrow y \in R_s$. Thus if $x \in J_e$ and card$(J_e) = \infty$, then $Q_{\langle e,x\rangle}$ requires attention. □

Corollary 2.6. *Let R be generated by a nonrecursive co-r.e. subset of a recursive basis. Then there is an r.e. subspace V with $V \oplus (R)^* = V_\infty$ such that if D is a decidable ($=$ complemented in the lattice of r.e. subspaces) subspace with $D \supset V$, then $D = V_\infty$.*

Corollary 2.7. *Let **A** be any co-r.e. nonrecursive nonisolic RET. Then **A** contains a co-r.e. basis of a subspace each basis of which is unsound.*

Proof. Use the technique in Theorem 2.3 via the V we constructed in Theorem 2.5. □

In view of these results the following is perhaps surprising:

Theorem 2.8. *There exists a co-r.e. isolated totally sound nonrecursive RET.*

Proof. We build $A = \bigcap_s A_s$ in stages. Our idea is to 'effectivise' Theorem 2.1. We satisfy, therefore,

P_e: $W_e \not\subset A$ if $\text{card}(W_e) = \infty$,
I_e: $\text{card}(A) \geq e$, and
N_e: If ϕ_e is 1–1, then either
(i) $\text{dom}\, \phi_e \not\supset A$,
(ii) $\phi_e(A)$ is not independent, or
(iii) there exists an r.e. independent set $B_e \supset \phi_e(A)$.

At each stage s, $b_0^s < b_1^s < \cdots$ will list in order the elements of A_s, and this will be a confinite subset of ω. The key difficulty in the construction is making a strategy for (iii). To this end, at each stage s, for each *active* N_e (viz, ones for which $\phi_{e,s}$ is 1–1), we shall be building a collection of independent sets $B_e^\sigma = \bigcup_s B_{e,s}^\sigma$, where σ is a binary string of length e. To co-ordinate the various N_e's, we shall use a sort of 'e-state', measured relative to independence over the various B_e^σ. Thus associated with each b_j^s for $j \geq e$, we shall have a binary string of length $e+1$, $\alpha = \alpha_0 \cdots \alpha_e$, called its *e-type* (with *guess* $\alpha_0 \cdots \alpha_{e-1}$). Here we define $\alpha_e = 0$, if $\phi_{i,s}$ is inactive. We define $\alpha_i = 0$, if $\phi_{i,s}(b_j^s)$ is independent over $B_{i,s}^\sigma$ where $\sigma = \alpha_0 \cdots \alpha_{i-1}$, the first i numbers of the guess of α. Otherwise, we set $\alpha_i = 1$. For $s > g$, $h \geq e$ we say the e-type of b_g^s is stronger than the e-type of b_h^s if the e-type of b_g^s is lexicographically less than the e-type of b_h^s.

We say N_e *requires attention at stage* $s+1$ *via* k *and* j if k, j are least such that $s \geq k > j \geq e$, and e is at least such that b_k^s has stronger e-type than b_j^s. We say P_e *requires attention via* x_e if e is least with x_e correspondingly least such that $x_e > b_e^s$ and $W_{e,s} \subset A_s$ and $x_e \in W_{e,s}$. In the construction we shall use the following 'computer science' convention: '$X \leftarrow X \cup Y$' means we rename $X \cup Y$ by X, and similarly for operations other than union. (This could be avoided by a more complicated notation, but we feel the meanings will be clear from context.)

Construction, stage $s+1$

Step 1. If no P_e for $e \leq s$ requires attention, go to Step 2. If P_e requires attention via x_e set $A_s \leftarrow A_s - \{x_e\}$. Renumber the b_i^s appropriately. Notice P_e is now (permanently) met. Go to Step 2.

Step 2. *Case* (i). If no N_e requires attention, go to Step 3, otherwise go to case (ii).

Case (ii). If e is least such that N_e requires attention via, say, k, j with $k > j$, let $A_s \leftarrow A_s - \{b_j^s, \ldots, b_{k-1}^s\}$. Rename the b_i^s appropriately, and go to case (i).

Step 3. For each $e \leq s$ perform the following substeps in order of e, for $e \leq j \leq s+1$.

Substep $j = e$. See if $\alpha^e = \alpha_0^e \cdots \alpha_e^e$, the e-type of b_e^s has $\alpha_e^e = 0$ and N_e is active. If so, rename $B_{e,s}^\sigma \leftarrow B_{e,s}^\sigma \cup \{\phi_{e,s}(b_j^s)\}$ where $\sigma = \alpha_0^e \cdots \alpha_{e-1}^e$ is the guess of α^e. Notice that this may change the e-type of later elements b_k^s with $k \geq e$. Go to substep $j = e+1$.

Substep $j > e$. See if $\alpha^j = \alpha_0^j \cdots \alpha_e^j$ the e-type of b_j^s, has $\alpha_j^e = 0$ and N_e is active. If so, rename $B_{e,s}^{\sigma} \leftarrow B_{e,s}^{\sigma} \cup \{\phi_{e,s}(b_j^s)\}$ where σ is the guess of α^j. If $j \leqslant s$, go to substep $j+1$. If $j = s+1$ and N_e requires attention for some $e \leqslant s$, go to Step 2. If $j = s+1$ and Step 2 doesn't pertain, set $A_{s+1} = A_s$ and for all σ of length $\leqslant s+1$, set $B_{e,s+1}^{\sigma} = B_{e,s}^{\sigma}$, then go to stage $s+2$.

To complete the construction, set $B_e^{\sigma} = \bigcup_s B_{e,s}^{\sigma}$ and $A = \bigcap_s A_s$.

Now we verify the construction. Evidently, once the P_j for $j < e$ stop requiring attention, b_e^s can only move to stronger e-types. Thus, for example, go to a stage where $b_i^s = b_i$ for all $i < e$ and each of the b_i have reached their final e-type. Then the guess of b_e can only change to improve the e-type of b_e^s. An induction therefore will show that $\lim_s b_e^s = b_e$ exists. It is easy to see by induction that all but finitely many b_j^s have the same e-type. Call this the *final e-type*, say $\alpha = \alpha_0 \cdots \alpha_e$, with guess $\alpha_0 \cdots \alpha_{e-1}$. If $e = 0$, then consider $\alpha_e = \alpha_0$. If $\alpha_e = 0$, then either N_e is inactive (and so met by fiat) or ϕ_e is 1–1. In this latter case we claim that it is easy to see that the construction ensures that B_0 is independent[3] and if $\alpha_e = 0$, then $(B_0)^* \supseteq \phi_e(A)$. If $\alpha_e = 1$, we claim that B_0 is finite: numbers may only enter B_0 if their e-type is 0 (relative to smaller numbers of A_s). These can only be replaced by other numbers with e-type 0. If $\alpha_e = \alpha_0 = 1$, then for all but finitely many j, the 0-type of $b_j^s = 1$ *permanently*. This means B_0 is finite, and so $\dim((\phi_0(A) \cup B_0)^*/(B_0)^*) < \infty$ and thus $\phi_e(A)$ is not independent.

For an induction suppose e is least such that N_e is not met. Let $\sigma = \alpha_0 \cdots \alpha_{e-1}$ be the *correct guess*, namely, the final $e-1$ type, and $\alpha = \alpha_0 \cdots \alpha_e$ the final e-type. If $\alpha_e = 0$ and N_e is active, we claim it is easy to see that B_e^{σ} is independent and r.e. as for the $e = 0$ case. Also $B_e^{\sigma *} \supset \phi_e(A)$ since almost all b_j^s attain e-type α. When they do, eventually Step 2 and Step 3 will ensure that (i) the b_j^s is put into B_e^{σ} and (ii) ones with lower priority guesses, or ones with the same guess, but of lower e-type will be removed from A. Finally, if $\alpha_e = 1$, then for the same reason as the $e = 0$ case, *as σ is the correct guess*, B_e is finite and $\dim((\phi_e(A) \cup B_e^{\sigma})^*/(B_e^{\sigma})^*) < \infty$ and so all the N_e are met, and the verification is complete. □

Remark. For the reader familiar with tree-style priority arguments (cf. Soare [35]), we could convert the above to a tree argument. At each node σ associated with N_e we would have a σ-strategy attempting to build an r.e. set $B_{(e)}^{\sigma}$ if $\sigma = (0)$ or $\sigma = \tau\hat{\ }0$ where τ is the guess according to the final e-type (and the action of the P_j of higher priority). Then the final e-type (and the action of the P_e), in our notation, would correspond to the true path of the construction. The author also tried a 'pinball' type argument where gates correspond to guesses, but in the end it appeared that the current presentation seemed the most perspicuous.

It is easy to see that the above argument codes so that we can build A above any given r.e. degree, in particular of degree $\mathbf{0}'$. It is also not too difficult to combine this result with a maximizing e-state construction to ensure that A is

[3] Only one version of B_0 is built since no 'guessing' is involved.

co-maximal. In this case the guess will correspond to the current idea concerning the j-types for $j < e$ and i-states for $i \leq e$ (thus, the guess is a $2e+1$-tuple). Finally we remark, but do not prove, that the argument above can be modified using Martin permitting and coding to produce A in each high r.e. degree. We do not know if there is any lattice theoretic restriction upon A, (for example, co-maximality or co-r-maximality) and neither do we know if one can produce A of nonhigh degree. In the latter case we (apparently) will need to come up with a new construction. If the only such A are in high degrees (and perhaps have certain lattice properties amongst the r.e. sets) this will be in some sense surprising, since it will indicate very much deeper connections between the lattice of r.e. sets, the lattice of r.e. subspaces, and RET's than we currently know.

We do know that such A *cannot* exist in every r.e. degree. The next theorem in fact produces (below any given nonzero r.e. degree), a (low) r.e. degree bounding no nontrivial co-r.e. totally sound RET.

Theorem 2.9. (a) *Let $\delta \neq 0$ be any nonzero r.e. degree. Then there exists an r.e. nonzero degree ρ with $\rho \leq_T \delta$ such that if B is any co-r.e. set with $B \leq_T \rho$, then either B is r.e. or B is recursively equivalent to an unsound independent set.*

(b) *In fact we can ensure that, in the latter case, B is recursively equivalent to a basis of an unsound subspace.*

Proof. (a) We build an r.e. set $A = \bigcup_s A_s$ in stages, to have the desired properties. We ensure that $A \leq_T \delta$ by simple permitting: thus, let D be r.e. of degree δ and let $f(\omega) = D$ be a 1–1 recursive enumeration of D. We put elements x into A_{s+1} only if $f(s) \leq x$. This ensures that $A \leq_T D$. To ensure that A is not recursive we meet

P_e: $\bar{A} \neq W_e$.

We satisfy the P_e by appointing *followers* x_e of P_e. Initially these are *unrealized* and become *realized* if $x_e \in W_{e,s}$. Followers remain appointed unless they are *cancelled*, through the action of higher priority requirements. We say P_e *requires attention* if e is least such that P_e has an (uncancelled) realized follower x_e which is *permitted*, that is, $f(s) \leq x_e$, or P_e has no unrealized follower and $W_{e,s} \cap A_s = \emptyset$.

For the negative requirements, we need some standard notations: $\Phi_{e,s}(D;x)$ denotes the result, if any, of performing s steps of the eth oracle machine with oracle D and input x. If this halts we write $\Phi_{e,s}(D;x)\downarrow$ and $\Phi_{e,s}(D;x)\uparrow$ otherwise. As usual we identify sets with their characteristic functions. We must satisfy

N_e: If $\Phi_e(A) = f$ (= a characteristic function) and f is total and co-r.e., then either

(i) f is r.e., or

(ii) f is recursively equivalent to an unsound independent set.

To satisfy the N_e, we split them into the infinitely many subrequirements $N_{\langle e,j\rangle}$

below. Recall that I_e denotes the eth r.e. independent set. We shall build a partial recursive 1–1 function γ_e in stages (if $\Phi_e(A)=f$). We shall in fact satisfy

$N_{\langle e,j\rangle}$: If $\Phi_e(A)=f$ and f is total and co-r.e., then either
(i) f is r.e., or
(ii) $\gamma_e(f)\nsubseteq I_j$.

In the construction to follow, associated with each $N_{\langle e,j\rangle}$ will be certain auxilliary functions, $m(\langle e,j\rangle,s)$ and $n(\langle e,j\rangle,s)$, marking the ends of a certain region used to attack $N_{\langle e,j\rangle}$. There will be a function $r(\langle e,j\rangle,s)$ denoting a restraint (on A) corresponding to $m(\ ,\ )$ and $n(\ ,\ )$ above. There will also be a set of *immovable* markers $\Gamma(\langle e,j\rangle,i)$ which are initially not marking anything, but when placed on some elements are never removed. Moreover, if $\Gamma(\langle e,j\rangle,i)$ is placed on x and y (say) at some stage s, and if a marker $\Gamma'(\langle e,j\rangle,k)$ is placed on some elements at some stage $s'>s$ then $k>i$. Thus the 'i' will serve as a record of 'attackers' for the $N_{\langle e,j\rangle}$.

The key point is that we consider *only co-r.e. f*. Thus we suppose that if, at any stage s, $\Phi_{e,s}(A_s;x)=0$, then $\forall t>s\ (\Phi_{e,s}(A_s;x)=0)$, since if f is co-r.e. then it can be so presented. Here we define our reductions to terminate at stage t, if at some stage s, $\Phi_{e,s}(A_s;x)=0$ and at stage $t>s$, $\Phi_{e,s}(A_s;x)=1$, by thereafter declaring $\Phi_{e,t'}(A_s;y)=0$ for all y and $t'\geqslant t$. Also, it is clear that we need only consider reductions such that $\Phi_{e,s}(A_s;x)\downarrow$ only if $\forall y<x\ (\Phi_{e,s}(A_s;x)\downarrow)$. Again this does no harm since we need only delay the appropriate output.

At any stage s, define $l(e,s)=\max\{x\colon \forall y\leqslant x\ (\Phi_{e,s}(A_s;y)\downarrow)\}$; and let $u(e,x,s)$ be the maximum element used in the computation of $\Phi_{e,s}(A_s;x)$ if this halts, with $u(e,x,s)$ undefined otherwise.

For clarity, we first give the construction in the case that F, the underlying field, is infinite. We will later give the more complicated construction in the case where F is finite. We say $N_{\langle e,j\rangle}$ *requires attention at stage* $s+1$ if $\langle e,j\rangle$ is at least such that either (a) or (b) below occurs:

(a) (i) $N_{\langle e,j\rangle}$ is inactive, and
(ii) $\exists x,y\in I_{j,s}-\{\gamma_{e,s}(x)\mid \Phi_{e,s}(A_s;x)=1\ \&\ x\leqslant l(e,s)\}$, such that
(iia) $I_{j,s}-\{\gamma_{e,s}(x)\mid \Phi_{e,s}(A_s;x)=1\ \&\ x\leqslant l(e,s)\}$ is independent over $(\{\gamma_{e,s}(x)\mid \Phi_{e,s}(A_s;x)=1\ \&\ x\leqslant l(e,s)\})^*$, and
(iib) $\{x,y\}$ is independent over $(\{\gamma_{e,s}(x)\mid \Phi_{e,s}(A_s;x)=1\ \&\ x\leqslant l(e,s)\}\cup \bigcup_{i<\langle e,j\rangle} B_{i,s})^*$, or

(b) (i) $N_{\langle e,j\rangle}$ is active, and
(ii) $\exists q\ [m(\langle e,j\rangle,s)<q<l(e,s)\ \&\ \Phi_{e,s}(A_s;q)=1]$, and
(iii) $I_{j,s}-\{\gamma_{e,s}(x)\mid \Phi_{e,s}(A_s;x)=1\ \&\ x\leqslant l(e,s)\}$ is independent over $(\{\gamma_{e,s}(x)\mid \Phi_{e,s}(A_s;x)=1\ \&\ x\leqslant l(e,s)\})^*$.

Priority Ranking: $N_0, P_0, N_1. P_1, \ldots$.

Construction

Stage 0. For all $\langle e, j\rangle \in \omega \times \omega$ declare $N_{\langle e,j\rangle}$ as inactive, set $A_0 = \emptyset$ and set $B_{k,0} = \emptyset$ for all $k \in \omega$.

Stage $s+1$. Adopt the case below appropriate to the requirement of highest priority.

Case (1). For $e \leqslant s$, no P_e or N_e requires attention. Do nothing.

Case (2). If P_e requires attention, and has no unrealized follower, appoint a large unrealized follower x of P_e. Namely let x be the least number exceeding all followers, restraints, uses, etc., at any stage $\leqslant s$. Otherwise, do nothing.

Case (3). If P_e requires attention via a permitted realized follower, let x_e be the least such, and set $A_{s+1} = A_s \cup \{x_e\}$. Now for $j \geqslant e$ cancel any followers of P_j. Cancel all restraints $r(k, s)$ for $k > e$, by setting $r(k, s+1) = r(e, s)$. For all $k > e$, also declare N_k as inactive, set $B_{k,s+1} = \emptyset$ for all such k, and otherwise do nothing.

Case (4). If $N_{\langle e,j\rangle}$ requires attention and is inactive, find the least pair $\{x, y\}$ pertaining to $N_{\langle e,j\rangle}$ and the least unused marker $\Gamma = \Gamma(\langle e, j\rangle, i)$ and mark both x and y by Γ. Set $m(\langle e, j\rangle, s+1) = l(e, s)$. Set $r(\langle e, j\rangle, s+1) = 1 + \max\{u, r(\langle e, j\rangle, s)\}$, where $u = \max\{u(e, x, s) \mid x \leqslant l(e, s)\}$. Declare $N_{\langle e,j\rangle}$ as active. Cancel all followers of P_k for $k \geqslant \langle e, j\rangle$ and declare all N_p for $p > \langle e, j\rangle$ as inactive, and set $B_{p,s+1} = \emptyset$ for all such p. Set $B_{\langle e,j\rangle,s+1} = \{x, y\}$ and otherwise do nothing.

Case (5). If $N_{\langle e,j\rangle}$ requires attention and is active, find the largest i such that $\Gamma(\langle e, j\rangle, i)$ is used, that is, marks a pair $\{x, y\}$. In this case, as $N_{\langle e,j\rangle}$ was not inactivated, we suppose $\{x, y\}$ is independent over

$$\left(\{\gamma_{e,s}(x) \mid \Phi_{e,s}(A_s; x) = 1 \ \&\ x \leqslant l(e, s)\} \cup \bigcup_{i<\langle e,j\rangle} B_{i,s}\right)^*.$$

(Of course, we verify this in the proof.) Thus, as F is infinite, we can find λ_1, λ_2 with $\lambda_i \neq 0$ and $\lambda_1 x + \lambda_2 y \notin \{\gamma_{e,s}(x) \mid \gamma_{e,s}(x)\downarrow\}$. Thus our action is to define $\gamma_{e,s+1}(q) = \lambda_1 x + \lambda_2 y$, set $B_{i,s+1} = B_{i,s}$ for $i \leqslant \langle e, j\rangle$, and declare $N_{\langle e,j\rangle}$ as inactive. For $k \geqslant \langle e, j\rangle$, we cancel all followers of P_k, set $B_{k+1,s+1} = \emptyset$, and declare N_k as inactive. Now set $n(\langle e, j\rangle, s+1) = l(e, s)$ and set $r(\langle e, j\rangle, s+1) = 1 + \max\{u, r(\langle e, j\rangle, s)\}$, where $u = 1 + \max\{u(e, x, s) \mid x \leqslant l(e, s)\}$. Finally, cancel all restraints $r(k, s)$ for $k > \langle e, j\rangle$ by setting $r(k, s+1) = r(\langle e, j\rangle, s+1)$. Otherwise, do nothing.

To complete the construction, set $A = \bigcup_s A_s$.

It is not too difficult to verify the construction (by induction). Go to a stage t where all N_i and P_i for $i < e$ cease to act. Then their corresponding restraints, etc., cease to change. In particular, for all $i < e = \langle k, j\rangle$, and for all $s \geqslant t$, $B_{i,s} = B_{i,t} = B_i$, say. Now suppose $\gamma_k(\Phi_k(A)) \subset I_f$, and $\Phi_k(A)$ is not r.e. Then provided γ_k is 1–1, $\gamma_k(\Phi_k(A))$ is not r.e. and in particular $I_f - \gamma_k(\Phi_k(A))$ is an infinite independent set. Thus there must, at some time $t' > t$, exist appropriate $\{x, y\}$ for (a) of the definition of $N_{\langle k,j\rangle}$ requiring attention. Now these will be marked by a

Γ and $N_{\langle k,j\rangle}$ will be declared active. By hypothesis, it will never be inactivated except through case (5). Moreover, it is easy to see that at the next expansion stage t'' (as we restrain appropriately), $\Phi_{k,t'}(A_s; z) = \Phi_{k,t''}(A_s; z)$ for all $z \leq l(k, t')$; and $\{x, y\}$ is independent over $(\{\gamma_{k,t''}(\Phi_{k,t''}(z)) \mid z \leq l(k, t'')\} \cup \bigcup_{i<\langle k,j\rangle} B_{i,t''})^*$. Now, we can find λ_1, λ_2 as in case (5), since there are infinitely many such pairs, and only finitely many members of $\{\gamma_{k,t''}(z) \mid \gamma_{k,t''}(z)\downarrow\}$. We then find such a pair. This will ensure that, as we now restrain $\{\gamma_e(z) \mid x \leq l(k, t'')\}$ for all stages $s > t''$ (since we restrain on the appropriate use of $\Phi_k(A)$), $r(\langle k, j\rangle, s)$ will have settled down, and $N_{\langle e,j\rangle}$ won't again require attention since $x, y \in I_j$ and some linear combination of x and y has occurred in $\gamma_k(\Phi_k(A))$. Evidently, a similar analysis also shows that γ_k is 1–1, since $\gamma_{k,s}(x)\downarrow$ only through the action of case (4) applied (by induction) to appropriately independent $\{x, y\}$. Finally, observe that independence is preserved through case (4) because of the convention discussed before the construction.

Once $N_{\langle k,j\rangle}$ has ceased to require attention and $r(\langle k, j\rangle)$ has reached its final value, followers appointed to P_e are never cancelled. If infinitely many are appointed, as usual, we get an infinite increasing list of unrealized followers that are never permitted, which would show that $D = \mathfrak{f}(\omega)$ is recursive. Therefore, only finitely many followers are appointed, and so P_e is similarly met. The result now follows by induction.

Now the case where F is finite. The problem comes in case (4), where we need to know that some linear combination of x and y is available. If $F = \mathrm{GF}(2)$, then the only such combination is $x + y$. Now although $\{x, y\}$ would still be independent over $\{\gamma_e(x) \mid \Phi_{e,s}(A_s; x) = 1 \ \&\ x \leq l(e, s)\}$, it may be the case that some $N_{\langle e,t\rangle}$ has previously assigned $\gamma_{e,s}(k) = x + y$, so that γ_e will *not* be 1–1. This can only happen if $x + y$ has been removed from $\{\gamma_{e,s}(x) \mid \Phi_{e,s}(A_s; x) = 1 \ \&\ x \leq l(e, s)\} = G_s$, say, since otherwise $I_{j,s} - G_s$ is not independent over G_s, and case (4) couldn't have applied after all. Of course, here we could define $\gamma_{e,t}(q) = x + y$, but this would show that γ_e is a partial recursive function whose *restriction* to $\Phi_e(A)$ is 1–1 (this latter part by the 'co-r.e.' convention). The solution is as follows. We wait until two new elements p, q occur, with $\Phi_{e,s}(A_s; p) = \Phi_{e,s}(A_s; q) = 1$ and with $p, q > m(\langle i, j\rangle, s)$. Now in case (4) we find a $z \in V_\infty$ such that $\{x, z + x + y\}$ is independent over

$$(I_{j,s} - \{x, y\} \cup \{\gamma_{e,s}(u) \mid \Phi_{e,s}(A_s; u) = 1 \text{ and } u \leq l(e, s)\})^*$$

and

$$z, z + x + y \notin \{\gamma_{e,s}(u) \mid \gamma_{e,s}(u)\downarrow\}.$$

Now set $\gamma_{e,s+1}(p) = z$ and $\gamma_{e,s+1}(q) = z + x + y$, set $B_{e,s+1} = \{x, y, z\}$, and otherwise proceed as before. Again, $I_{j,s} \cup \gamma_{e,s}(\Phi_{e,s}(A_s)$ cannot be thereafter independent. Here, the obvious slight change is needed in clause (iii) of both (a) and (b) of $N_{\langle e,j\rangle}$ requiring attention. The remaining details are similar.

Finally, we consider part (b) of the statement of the theorem. We must now replace the definition of $N_{\langle e,j\rangle}$ by:

$N'_{\langle e,j\rangle}$: If $\Phi_e(A)=f$ and f is co-r.e. and total, then
either (i) f is r.e.
or (ii) $((\gamma_e(f))^* \cap I_f)^* \neq (\gamma_e(f))^*$.

The idea is very similar to that for part (a). Consider, for simplicity, F infinite. Basically we wait until a stage occurs when $x, y \in I_f$ with $\{x, y\}$ independent over $(\gamma_e(f))^*$ and place a linear combination of x and y into $\gamma_e(f)$, whilst putting $\{x, y\}$ into B_e to hold them out of $\gamma_e(f)$ with priority e. ($N_{\langle e,j\rangle}$ can only require attention if $I_{j,s}-(\gamma_e(f_s))^*$ is independent over $(\gamma_e(f_s))^*$. We leave the remaining details to the reader. (Cf., for example [27].) □

3 General settings

In this section we discuss RET's, soundness, etc. for a general class of algebraic structures. We would like to have a general idea of morphisms for such structures, so the setting we shall consider will be an *Effective Closure Algebra.* This is a universal algebra $\mathfrak{M}=\langle M, R\rangle$[4] such that the algebraic closure operator $\mathrm{cl}=\mathrm{cl}_R$ induced by R satisfies

(i) (For $A\subset M$), $A\subset \mathrm{cl}(A)$,
(ii) $\mathrm{cl}(A)=\mathrm{cl}(\mathrm{cl}(A))$,
(iii) $x\in \mathrm{cl}(A)$ implies there exists a finite $A'\subset A$ with $x\in\mathrm{cl}(A')$,
(iv) $A\subset B$ implies $\mathrm{cl}(A)\subset\mathrm{cl}(B)$, and
(v) ('Local computability') if $y, x_1, \ldots, x_n\in M$, then there is an effective procedure to determine whether or not $y\in\mathrm{cl}(x_1,\ldots,x_n)$, (uniformly in the index of the tuple $\langle y, x_1, \ldots, x_n\rangle$).

We say $\mathfrak{M}$ is an effective *Steinitz closure algebra* (cf. [16], [22]), if $\mathfrak{M}$ also satisfies

(vi) (Exchange) $x\in\mathrm{cl}(A\cup\{y\})-\mathrm{cl}(A)$ implies $y\in\mathrm{cl}(A\cup\{x\})$.

These axioms are obviously satisfied by many algebraic systems. Examples include groups with solvable generalized word problems (with cl a subgroup operator), abelian groups of the same type, locally computable rings (where cl, the closure operator, is taken as either an ideal operator or subring operator), fields, boolean algebras, and various types of orderings. For a fuller discussion of these and other applications we refer the reader to [30] and [17]. Examples of Steinitz closure algebras are (ω, cl): sets with $\mathrm{cl}(A)=A$; $(V_\infty, {}^*)$; (F_∞, cl): where here F_∞ is a fixed effective algebraically closed field and cl is an effective algebraic closure operator (cf. [16]); (V_∞, Kl): where $y\in\mathrm{Kl}(x_1,\ldots,x_n)$ means $\exists\lambda_1,\ldots,\lambda_n$

[4] That is, a set M with a collection of finitary operations R. In an effective setting, M is recursive and is identified with ω.

with $\sum \lambda_i = 1$ and $y = \sum \lambda_i x_i$; and 'homogeneous' intersection subsystems (cf. [16]).

In the Steinitz setting, the idea of soundness is generalized in the obvious way. One can then perform analogues of many of the constructions from $(V_\infty, *)$, because in such a system if X and Y are independent sets, then any bijection between X and Y extends to an isomorphism between $\text{cl}(X)$ and $\text{cl}(Y)$. Moreover, if the bijection is recursive, so is the extension. One then may extend our results (and others) via the following classification results of Downey and Remmel. First we need two definitions:

Definition. Let $\mathfrak{M}$ be an effective Steinitz closure algebra, then

(a) We say $\mathfrak{M}$ is *n-semiregular* (Downey and Remmel [16]), if there is an $n > 0$ such that

(i) $\forall m < n\ (\{x_1, \ldots, x_m\}$ independent $\rightarrow (\text{cl}(x_1, \ldots, x_m) = \bigcup_{1 \leq i \leq m} \text{cl}(x_i))$,

and

(ii) $\forall t \geq n\ (\{x_1, \ldots, x_t\}$ independent $\rightarrow$

$$\exists z\ (z \in \text{cl}(x_1, \ldots, x_t) - \text{cl}(x_1, \ldots, x_{t-1}) - \text{cl}(x_2, \ldots, x_t)).$$

(b) We say $\mathfrak{M}$ is *autonomous* (cf. Baldwin [4]) if, for all $n \in \omega$, if $\{x_1, \ldots, x_n\}$ is independent then $\text{cl}(x_1, \ldots, x_n) = \bigcup_{1 \leq i \leq n} \text{cl}(x_i)$.

Theorem 3.1 (Downey and Remmel [16]). *If $\mathfrak{M}$ is any Steinitz closure algebra, then either $\mathfrak{M}$ is autonomous, or there is an $n > 0$ for which $\mathfrak{M}$ is n-semiregular.*

Theorem 3.2 (Baldwin [4]). *If $\mathfrak{M}$ is autonomous then the map $x_i \rightarrow \text{cl}(x_i)$ induces a recursive isomorphism from subsets of ω to closed subsets of $\mathfrak{M}$. In particular, the lattice of (r.e.) closed subsets of $\mathfrak{M}$ is recursively isomorphic to the lattice of (r.e.) sets.*

Armed with the above, we can extend our results to Steinitz closure algebras. We have:

Theorem 3.3. *Let $\mathfrak{M}$ be any effective Steinitz closure algebra. Then*

(i) *Theorem* 2.1, *Corollary* 2.2 *and Theorem* 2.8 *hold for $\mathfrak{M}$, and*

(ii) *Either $\mathfrak{M}$ is autonomous (in which case we are essentially dealing with the set case, and so every independent set is sound), or there are $2^{\aleph_0}$ unsound RET's which may be constructed in a similar way to Theorem* 2.3.

Proof. (i) is easy and is left to the reader. For (ii) we need to observe that the results of Downey [11] and Nerode and Remmel [27] show that each fully co-r.e. closed set has an r.e. supermaximal independent complement. The rest is the same via Theorems 3.1 and 3.2. □

We can also show:

Theorem 3.4. *Suppose $\mathfrak{M}$ is a n-semiregular effective Steinitz closure algebra. Then the analogues of Theorems* 2.3–2.7 *and Theorem* 2.9 *hold.*

Sketch of Proof (of, for example, the analogue of Theorem 2.9). We only sketch the proof as it relies on techniques to be found in, say, Nerode and Remmel [27]. Briefly, we use the case (of 2.9) where F is finite, and observe that once $\gamma_{e,s}(f_s)$ has at least n elements, then there is an r with

$$\begin{aligned} r \in \mathrm{cl}(\gamma_{e,s}(u), z, x, y \mid \Phi_{e,s}(A_s; u) = 1 \;\&\; u \leqslant l(e, s)) \\ - \mathrm{cl}(\gamma_{e,s}(u), z \mid \Phi_{e,s}(A_s; u) = 1 \;\&\; u \leqslant l(e, s)) \\ - \mathrm{cl}(\gamma_{e,s}(u), x, y \mid \Phi_{e,s}(A_s; u) = 1 \;\&\; u \leqslant l(e, s)). \end{aligned}$$

with $\{z, r\}$ independent over $\mathrm{cl}(\gamma_{e,s}(u))$, where here r assumes the role of $z + x + y$. For each such z there is an r, and there are therefore infinitely many possibilities for the pair $\{z, r\}$. We then find one to keep $\gamma_{e,s}$ 1–1, and proceed in a manner analogous to that of the proof of Theorem 2.9. The exchange property ensures that this (temporarily) meets $N_{\langle e,j \rangle}$. For suppose $I_j \supset \gamma_e(A)$ and z, u are not later removed. Then I_j contains x, y, r, z, and that segment $\{\gamma_e(u)\}$ of $\gamma_e(A)$ which it has used to find r. (Recall that all lower priority followers were cancelled, and restraints were raised above the uses at this length.) Assuming we are successful with this restraint, we see that, by exchange, $\mathrm{cl}(I_j - \{r\}) = \mathrm{cl}(I_j)$, and so I_j could not be independent. Now a finite injury argument shows one attack eventually works. For further details of this type of modification, see [27] or [10]. □

We remark that various other types of result generalize, (cf. [22]). However, we point out that not every result will. For example if we generalize the techniques of Dekker [19] or Hamilton [24], we can show that the 'sound' analogue of $\dim(V + W) \leqslant \dim(V) + \dim(W)$ holds for sound subspaces where 'dim' means the RET of a sound basis, in the spirit of Dekker [19]. Of course we *cannot* prove

$$\dim(V + W) = \dim(V) + \dim(W) - \dim(V \cap W)$$

in general, since this isn't classically true of even finite transcendence degrees in (F_∞, cl) (cf. [3]). Sometimes, results do generalize, but require new proofs. In [16], Downey and Remmel proved an analogue of Shore's [33] automorphism base for the lattice $L(\mathfrak{M})$ of r.e. closed sets for any Steinitz closure algebra. However, the proof splits into the sets (autonomous) case dealt with by Shore, and the n-semiregular one which required a new splitting theorem (one due to Ash and Downey), and the behaviour of complemented members of $L(\mathfrak{M})$ under automorphisms. (See [16] for details.)

Finally, we point out that for some results we simply don't know whether they generalize or not. One notable example of this is any situation where we need to know the exact nature of the 'morphisms' we are dealing with. Take (F_∞, cl) for instance. How many automorphisms does $L(F_\infty)$ have? What operations on F_∞ may be used to induce all the automorphisms of the lattice of subfields? Another example is lack of knowledge of the 'combinatorial decision theory' for such systems. Here, for instance, is it true that in any effective Steintiz closure algebra there is a uniformly effective procedure to decide the dimension of $\mathrm{cl}(\{x_1, \ldots, x_n\}) \cap \mathrm{cl}(\{y_1, \ldots, y_n\})$? Recently John Rosenthal provided an affirmative answer for (F_∞, cl).

For a general effective closure system, there may not be a notion of basis etc. We can however, define soundness of generator sets of various types. Let $X = \{x_1, \ldots, x_n, \ldots\}$. We say X is a *w-sequence* if for all m, $x_m \notin \mathrm{cl}(x_i \mid i < m)$; we say X is an *s-sequence* if for all m, $x_m \notin \mathrm{cl}(x_i \mid i \neq m)$; and X is a *k-sequence* if for all disjoint finite subsets Y, Z of X, $\mathrm{cl}(Y) \cap \mathrm{cl}(Z) = \mathrm{cl}(\emptyset)$. Notice here there are various definitions of RET's of these sequences. For example if X and Y are w-sequences we might define $X \cong_{\mathrm{W}} Y$ to mean $X \cong Y$ (as sets) via ρ, a recursive bijection which extends to a map $\rho^* : \mathrm{cl}(X) \xrightarrow{\cong} \mathrm{cl}(Y)$. There are other possibilities (cf. for example [30]). We have:

Theorem 3.4. (i) *Let $\mathfrak{M}$ be an effective closure system. Then there exist $2^{\aleph_0}$ set RET's not containing an unsound w-, s- or k-sequence. A similar analogue holds for Corollary* 2.2.

(ii) *Suppose further that $\mathfrak{M}$ contains an infinite w-sequence (respectively) s-sequence, k-sequence), then the above result is true with set RET's replaced by w-RET's (respectively s-RET's, k-RET's).*

Proof. For (i), directly generalize 2.1 and 2.2. For (ii), start with the appropriate sequence, and always add elements of the sequence. □

For the other results we may need additional axioms: One example is

Axiom I. In $\mathfrak{M}$, if X is any infinite w-sequence and Y is any finite set, then $\mathrm{cl}(X) \not\subseteq \mathrm{cl}(Y)$.

Theorem 3.5. *Suppose $\mathfrak{M}$ satisfies Axiom* I. *Then the analogue of Theorem* 2.8 *holds for $\mathfrak{M}$.*

Proof. Use the proof of Theorem 2.8, say, for w-sequences. The 'punch-line' of the proof is where we claim that if $\alpha_e = 1$ and σ is the correct guess so that $\alpha = \sigma^\wedge\alpha_e$ is the final-type, then since $\dim((\phi_e(A) \cup B_e^\sigma)^*/(B_e^\sigma)^*) < \infty$, $\phi_e(A)$ cannot be independent. Axiom I allows us to replace this reasoning. Suppose α as above. This means for all but finitely many b_j, $b_j \in \mathrm{cl}(B_e^\sigma)$. Suppose $\phi_e(A)$ is a w-sequence. Then $\phi_e(A)$ (actually $\phi_e(A')$ where $A-A'$ is finite), is an infinite w-sequence and yet each member of $\phi_e(A)$ is in $\text{cl}(B_e^\sigma)$ which is finitely generated (directly contradicting Axiom I.) □

We leave the reader to find other examples of 'sufficiency' axioms to produce (for example) unsound RET's (one axiom: there exists an infinitely w-generated closed set generated by $2^{\aleph_0}$ w-sequences). We feel these types of conditions are reasonably easy to find. A much more difficult question is finding necessary and sufficient conditions for certain features to exist. For example, Theorem 3.5 applies to an infinitely recursively generated free *abelian* group with cl the subgroup operator, to give co-r.e. totally sound RET's. The result, however, is also true for free subgroups of $\langle X, Y \mid - \rangle$, that is there is a co-r.e. isolated RET of a sound collection of free generators of a free subgroup which is totally sound. Notice that in the latter setting, evidently, Theorem 3.5 doesn't apply, but we can modify the proof of 2.8 in a similar way so that the result nevertheless holds. We remark that the settings which we feel portray the problems encountered most graphically are the lattices of boolean ideals of the free boolean algebra, ideals in $Z[x_1, x_2, \ldots]$, and free subgroups of $\langle X, Y \mid - \rangle$. Some of these are analysed in [13] and [17].

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