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MAXIMAL THEORIES

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1. Introduction

In this paper we analyse the way r.e. theories relate to one another, and in particular how they behave under extensions. Our viewpoint will be to consider those r.e. theories (of propositions) based on a fixed recursive set of literals $\{P_i \mid i \in \omega\}$, and in doing so we could also consider them as proper filters in the free boolean algebra with recursive meet ($\wedge$) join ($\vee$) and identity ($=$) relations. One of the fundamental observations concerning decidability of r.e. theories is that there are *essentially undecidable* theories, that is, theories with no (complete) decidable extensions. As is well known, a simple example of such a theory is obtained by considering the theory generated by $\{P_i \mid i \in A\} \cup \{\bar{P}_j \mid j \in B\}$ where A and B are r.e. recursively inseparable sets. This result was sharpened considerably by Martin and Pour-El [14] who showed that one could (using priority methods) find a pair of r.e. sets A, B as above such that T was essentially undecidable, and every r.e. theory T' extending T was a principal extension of T. In a sense this could be viewed as a 'maximal' r.e. theory in the sense that although it's set of extensions is classically 'thick', it's set of effective extensions is 'thin'. Our broad purpose in this paper is to investigate maximal r.e. theories: what maximality may be interpreted as in this lattice, and how other theories relate to a maximal theory.

Our starting point is the Martin–Pour-El result cited above. This example suggests a few concepts which we shall analyse in this paper. We say an r.e. theory is *well generated* if it is generated by a pair of sets $\{P_i \mid \in A\}$, $\{\bar{P}_j \mid j \in B\}$. We say an r.e. theory T has *few r.e. extensions* if T is essentially undecidable and every r.e. extension of T is a principal extension of T. We shall call an r.e. theory T a *Martin–Pour-El theory* if it is both well generated and has few r.e. extensions. Many questions suggest themselves: If T has few r.e. extensions is T contained in a Martin–Pour-El theory? Do r.e. Martin–Pour-El theories or theories with few r.e. extensions exist in every nonzero r.e. degree? Is every r.e. essentially undecidable theory contained in Martin–Pour-El theory? — a theory with few r.e. extensions?

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Here we show that all the questions above have negative answers. We shall summarize our results by section, but before doing so we mention a weaker type of maximality condition which turned out to be important to our investigations: We say an r.e. theory T has *relatively few r.e. extensions* if T is essentially undecidable and every r.e. theory containing T has a common principal extension with T. If T is also well generated, we say T is *weakly Martin–Pour-El.*

In Section 2 we dispose of the preliminaries, definitions and notations, etc.

In Section 3 we analyse Martin–Pour-El theories. Suppose $\{P_i \mid i \in A\}$, $\{\bar{P}_j \mid j \in B\}$ generate an r.e. theory T. We show that:

Corollary 3.4. *There exists an r.e. Martin–Pour-El theory with $A \cup B$ effectively simple (so $T \equiv_T \mathbf{0}'$).*

Theorem 3.5. *If T is weakly Martin–Pour-El, then $A \cup B$ is hypersimple.*

Theorem 3.7. *There exists an r.e. Martin–Pour-El theory such that $T \equiv_T A \cup B$ is of low degree (i.e., $T' \equiv_T \emptyset'$).*

In fact we generalize this to show

Theorem 3.9. *Let n be given. There exists an r.e. Martin–Pour-El theory T such that $T \oplus W_n^T \equiv_T \mathbf{0}'$.*

In Section 4 we analyse the degrees which do and do not contain Martin–Pour-El theories. The main result is:

Theorem 4.4. *Below any r.e. nonzero degree $\boldsymbol{\gamma}$, there exists a nonzero r.e. degree $\boldsymbol{\delta} \leq \boldsymbol{\gamma}$ that bounds no degree containing an r.e. Martin–Pour-El theory.*

This is particularly surprising in view of the facts that (by Section 3) there are Martin–Pour-El theories in $\text{low}_{n+1} - \text{low}_n$ and $\text{high}_{n+1} - \text{high}_n$ for all n and the following two results

Theorem 4.1. *There exists an r.e. weakly Martin–Pour-El theory in each nonzero r.e. degree.*

Theorem 4.5. *There exists an r.e. Martin–Pour-El theory such that* $\deg(T)$ *is high and incomplete and, $A \cup B$ is an r.e. maximal set.*

However, 4.1 should be contrasted with

Theorem 4.3. *Each nonzero r.e. degree contains an r.e. weakly Martin–Pour-El theory that is not Martin–Pour-El.*

In Section 5 we show how to generalize our results to theories which are not well generated. The main results here are

Theorem 5.1. *There is an r.e. theory T with few r.e. extensions contained in no r.e. Martin–Pour-El theory.*

Theorem 5.3. *Below any nonzero r.e. degree, there is a nonzero r.e. degree $\boldsymbol{\delta}$ which bounds no r.e. theory with few r.e. extensions.*

Theorem 5.4. *If T is weakly Martin–Pour-El, but not Martin–Pour-El, then T is an essentially undecidable theory contained in no r.e. theory with few r.e. extensions.*

Finally, Section 6 deals with maximality amongst r.e. theories with decidable extensions, that is, the 'non-essentially undecidable case'. Indeed we analyse the lattice $L(D)$ of r.e. subtheories of a fixed complete decidable theory D. Here, the natural way to analyse this is under congruence $=^*$ where for T_1, T_2 subtheories of D, we define $T_1 =^* T_2$ if they have a common principal extension in $L(D)$. With this definition we may analyse $L(D)$ in a way similar to other lattices of r.e. substructures (see for example [15]), since $=^*$ is an equivalence relation (it is obviously *not* in general). We show how one may define maximal, etc., with this in quite a natural way. For example, T is *maximal* in $L(D)$ if $T \neq^* D$ and for all $T' \in L(D)$, if T' is an extension of T, then either $T' =^* T$ or $T' =^* D$. The existence of such maximal theories is not surprising in view of the fact that we can find an r.e. theory T such that $L(T, D)$ (= the lattice of r.e. subtheories of D which extend T) is recursively isomorphic to the lattice $L(\omega)$ of r.e. sets. (This result also shows that the first-order theory of the lattice of r.e. theories is undecidable since Hermann [9, 7, 8] has shown $\mathrm{Th}(L(\omega))$ is undecidable.) Finally, we show that the study of $L(D)$ is therefore richer than that of $L(\omega)$ by studying 'nonextendibility amongst r.e. axioms' for r.e. subtheories of D, this being a structural feature of subtheories which certainly cannot occur in the lattice of r.e. sets.

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2. Notations and terminology

As we remarked earlier, it is useful to consider theories as filters of the atomless boolean algebra $\mathbf{Q}$ on the fixed set of generators $\{P_i \mid i \in \omega\}$. If $A \subset \mathbf{Q}$ we let A^* denote theory (filter) generated by A. As usual, we say A is principal if $A = \{a\}^*$ for some $a \in \mathbf{Q}$, and for theories A_1 and A_2 we say A_1 is *principal over* A_2 if there exists $a \in \mathbf{Q}$ such that $A_1 = (A_2 \cup \{a\})^*$. We shall write $(A, a)^*$ for $(A \cup \{a\})^*$. We remark that as usual a theory A is consistent if $A \nvdash 0$ (or $0 \notin A$).

We enumerate as $\{W_e\}_{e \in \omega}$ the r.e. consistent theories, and $\{F_e\}_{e \in \omega}$ the *well generated* theories. These may be considered as enumerated by letting (A_e, B_e) be an enumeration of the r.e. disjoint subsets of ω and setting $F_e = (P_i, \bar{P}_j \mid i \in A_e \,\&\, j \in B_e)$. We denote by w_e the e-th r.e. set. We consider $W_{e,s} = \{x \mid x \in (w_{e,s})^* \,\&\, x \leqslant \max\{s, y \mid y \in w_{e,s}\}\}$, where $\{w_{e,s}\}$ is some fixed enumeration of w_e. $\langle\, , \rangle$ will denote some fixed pairing of ω and $\langle, \ldots, \rangle$ will denote $\langle\langle\, , \rangle\, , \rangle \cdots \rangle$. Φ_e and Ψ_e will be used to oracle machines and we write $\Phi_{e,s}(M; z)$ for the result, if any, of computing s steps in the computation of the e-th oracle with oracle M and input z. If this converges we write $\Phi_{e,s}(M; z)\downarrow$, and $\Phi_{e,s}(M; z)\uparrow$ otherwise. We write $\Phi_e(M; z)\downarrow$ if $\exists s\, (\Phi_{e,s}(M; z)\downarrow)$. We use the standard *use function* as follows:

$$u(k, A, x) = \begin{cases} \mu y \quad (y \geqslant x \text{ and the computations for } \Phi_k(A; x)\downarrow \text{ and} \\ \qquad\qquad \Phi_k(A \upharpoonright y; x)\downarrow \text{ are identical}), \\ \text{undefined}, \quad \text{otherwise}, \end{cases}$$

where $A \upharpoonright y = \{z \in A \mid z \leqslant y\}$. If we are equipped with an enumeration $\{A_s\}$ of A, then we write $u(k, A, s, x)$ for $u(k, A_s, x)$. We identify sets where appropriate with their characteristic functions, and similarly, if, say $\forall x\, (\Phi_k(A; x) = B(x))$ we write $\Phi_k(A) = B$.

A few further remarks are perhaps in order. When treating theories it suffices to consider only elements of the form $\bigvee \varepsilon_i P_i$ where $\varepsilon_i P_i$ will *always* denote P_i or $\bar{P}_i$. This is because $T \vdash x \wedge y$ if and only if $T \vdash x$ and $T \vdash y$. Thus, writing each element in conjunctive normal form, it becomes clear we need only consider elements in $\bigvee \varepsilon_i P_i$ form.

If A and B are r.e. theories (consistent) we write $A =^* B$ if there exists x such that both $(A, x)^*$ and $(B, x)^*$ are consistent and $(A, x)^* = (B, x)^*$. Thus the definition for A being weakly Martin–Pour-El reads: for all B extending A, $B =^* A$.

All other notation is essentially standard, unless specifically stated, and may be found in [19] or [20].

3. Lattice properties

In this section we establish various lattice properties associated with classes of well generated theories, and in particular Martin–Pour-El type theories. We begin with the fundamental construction of a Martin–Pour-El theory. We feel that as this construction forms a basis of many later ones, we should give it as an aid to the reader. The proof and result are due to Martin and Pour-El [14].

Theorem 3.1. *There exists an r.e. Martin–Pour-El theory T.*

Proof. Recall that W_e denotes the e-th r.e. theory, and is given by enumerating elements of the form $\bigvee \varepsilon_i P_i$. At each stage s, we shall have constructed T_s and add a (possibly empty) subset of the free generators $\{P_i \mid i \in \omega\}$ (or their negations) to T_s to form T_{s+1} so that $T = \bigcup_s T_s$ has the desired properties. We assume the P_i are ordered via $P_0 < P_1 < \cdots$, and at each stage s we define the list $B_{0,s} < B_{1,s} < \cdots$ to list the set $\{P_i \mid P_i, \bar{P}_i \notin T_s\}$. We satisfy the following requirements (for $e \in \omega$):

$$R_e:\quad 0 \notin (T, W_e)^* \to \exists x\, ((T, x)^* = (T, W_e)^*).$$

We shall construct x as $x = \bigwedge_{\text{finite}} x_i$, where the x_i are elements of a certain 'witnessing' set Q_e, and we shall construct in stages as $Q_e = \lim_s Q_{e,s}$. We say that R_e *requires attention at stage* $s+1$ if e is least such that there exists $y \in W_{e,s+1}$ such that $y \notin (T_s, Q_{e,s})^*$ and $0 \notin (T_s, W_{e,s})^*$. If y is least for e we say R_e requires attention *via* y. Our attacks on R_e must clearly ensure that T is consistent incomplete and perfect, that is we meet

$$N_e:\quad \lim_s B_{e,s} = B_e \text{ exists.}$$

Construction

Stage 0. Set $T_0 = Q_{e,0} = \emptyset$ and so $B_{i,0} = P_i$ for all $i \in \omega$.

Stage $s+1$. Do nothing unless R_e requires attention for some $e \leqslant s$. If R_e requires attention via y, define

$$L(e, s, y) = \{\overline{\varepsilon_i B_{i,s}} \mid \varepsilon_i B_{i,s} \text{ occurs in } y \text{ for } i > e\}.$$

Now set $T_{s+1} = (T_s, L(e, s, y))^*$ and $Q_{e,s+1} = Q_{e,s} \cup \{y\}$, and say y is *acted on via* e at stage $s+1$. □ End of Construction

Lemma 3.2. *If y is acted on via e at stage $s+1$, then there exists x a boolean combination of* $\{B_{0,s}, \ldots, B_{e,s}\}$ $(=\{B_{0,s+1}, \ldots, B_{e,s+1}\})$ *such that* $T_{s+1} \vdash y \leftrightarrow x$.

Proof. Without loss of generality we may write

$$y = \bigvee_{i \leqslant e} \varepsilon_i B_{i,s} \vee \bigvee_{i > e} \varepsilon_i B_{i,s} \vee \bigvee_{\varepsilon_i P_i \in T_s} \varepsilon_i P_i \vee \bigvee_{\overline{\varepsilon_i P_i} \in T_s} \varepsilon_i P_i.$$

That is $y = x \vee z \vee m \vee n$, say. Since $\vdash x \to y$, it suffices to show that $T_{s+1} \vdash y \to x$. Notice that if $m \neq 0$, then as $\vdash m \to y$ and as $m \in T_s$, $y \in T_s$ and so R_e cannot require attention via y. Therefore $m = 0$. Now as $\bar{n} \in T_{s+1}$ and $\bar{z} \in T_{s+1} - T_s$ (by construction), it follows that $T_{s+1} \vdash y \to x$ as required. □

Lemma 3.3. *Q_e is finite and* $\lim_s B_{e,s} = B_e$ *exists.*

Proof. By induction let t be least with $B_{i,s} = B_{i,t} = B_i$ for all $s \geq t$ and $i \leq e$. We claim $B_{e+1,s}$ may change at most finitely often subsequently (and here Q_e is finite). When R_e requires attention via y at stage $s+1$ ($s+1 > t$), $T_s \nvdash y$. By Lemma 3.2, $T_s \vdash y \leftrightarrow x$, where x us derived above as a boolean combination $B_0, \ldots, B_e$. Clearly then $(T_{s+1}, Q_{e,s+1})^* \vdash x$. Notice $(T_s, Q_{e,s})^* \nvdash x$ for if otherwise as $\vdash x \to y$ (by construction), it would follow that $y \in (T_s, Q_{e,s})^*$, contradiction to R_e requiring attention. That is, each time R_e requires attention after stage t, we must choose a new boolean combination of $B_0, \ldots, B_e$ of which there are only 2^{2^e} such. Notice it is immediate that all the R_e are met. □

A moment's thought reveals that if $T = (P_i, \bar{P}_j;\ i \in A, j \in B)^*$ is constructed using the *above* construction, then $A \cup B$ is effectively simple, that is, there is a recursive function f such that if $w_e \cap (A \cup B) = \emptyset$, then $\text{card}(w_e) < f(e)$. Thus we have,

Corollary 3.4. *There exists a Martin–Pour-El theory T with $T \equiv_T \mathbf{0}'$.*

Proof. It is well known (see e.g. [19]) that effectively simple sets are complete. Notice that if T is well generated, then $T \equiv_T A \cup B$ as follows: given x, put x into conjuctive normal form, namely $x = \bigwedge_{i \in G} x_i$ say where $x_i = \bigvee_{j \in I_i} \varepsilon_{j_i} P_j$. Then

$$x \in T \leftrightarrow \forall i \in F\ \exists j \in I_i\ ((j \in A \text{ if } \varepsilon_{j_i} P_j = P_j) \text{ or } (j \in B \text{ if } \varepsilon_{j_i} P_j = \overline{P_j})). \quad \square$$

We shall continue to implicitly use $T \equiv_T A \cup B$ henceforth. The sets A, B form a pair of recursively inseparable sets of a special type and we say r.e. sets A, B are a *maximal pair* if $A \cap B = \emptyset$, $\text{card}(\omega - (A \cup B)) = \infty$ and if C and D are r.e. disjoint sets with $C \supset A$ and $D \supset B$, then $\text{card}(C - A) < \infty$ and $\text{card}(D - A) < \infty$. It would seem reasonable to conjecture that if T is generated by a maximal pair, then T is at least weakly Martin–Pour-El. The next pair of results show that this is not the case. One interpretation of this is that 'maximality' in the well generated theories does not imply the same in the general theories. According to [14] the following result for Martin–Pour-El theories was discovered by A. Boxer (unpublished).

Theorem 3.5. *If $T = (P_i, \overline{P_j} \mid i \in A, j \in B)^*$ is a weakly Martin–Pour-El r.e. theory, then $A \cup B$ is hypersimple.*

Proof. Suppose $A \cup B$ is not hypersimple. Then there exists an infinite recursive sequence of pairwise disjoint finite sets $\{D_n\}_{n\in\omega}$ such that $\forall n\,(D_n \cap \overline{(A\cup B)} \neq \emptyset)$. Let $E_n = D_{2n} \cup D_{2n+1}$. Then

$$\forall n \left(\operatorname{card}\left(E_n - \left((A\cup B) \cup \bigcup_{m\neq n} E_m\right)\right) \geqslant 2\right).$$

We shall define $G = \bigcup_s G_s$ in stages, such that $G \supset T$ and $G \neq {}^*T$. At each stage s, $x_s = \bigvee_{i\in E_s} P_i$ is put into $G_{s+1} - G_s$. We ensure that some $y_e = \lim_s y_{e,s}$ is in G such that $y_e = \bigvee_{i\in F_e} P_i$ where $F_e \subset E_e$, and $\operatorname{card} F_e \geqslant 2$ and $\forall m\,(F_m \cap [(A\cup B) \cup \bigcup_{n\neq m} F_n] = \emptyset)$ from which it will follow that $G \neq {}^*T$. We build $F_e = \bigcup_s F_{e,s}$ in stages.

We say that e *requires attention* at stage $s+1$ if $\exists j \in E_e$ such that P_j occurs in $y_{e,s}$ and $j \in A_s \cup B_s$. (We tacitly assume $T_s = (P_i, \overline{P_j} \mid i \in A_s, j \in B_s)^*$.

Construction

Stage 0. Set $G_0 = \emptyset$ and define $y_{e,0} = x_e$ all $e \in \omega$.
Stage s + 1. Step 1. If no $e < s$ requires attention define for all $i < s$, $r_{i,s} = y_{i,s}$ and $K_{i,s} = F_{i,s}$ and finally $M_s = G_s$, and go to Step 3.

Step 2. If e requires attention with e least, find the least $j = j_s$ in $F_{e,s}$ such that P_j occurs in $y_{e,s}$ and $j \in (A_s \cup B_s)$. (By induction we assume $y_{e,s} = \bigvee_{i\in F_{e,s}} P_i$). Now define

$$r_{e,s} = \bigvee_{\substack{i\in F_{e,s}\\ i\neq j_s}} P_i, \qquad K_{e,s} = F_{e,s} - \{j_s\} \quad \text{and} \quad M_s = G_s \cup \{r_{e,s}\}.$$

Now for $i < s$ and $i \neq e$ define $r_{i,s} = y_{i,s}$ and $K_{i,s} = F_{i,s}$.

Step 3. Finally, set for $i \leqslant s$,

$$F_{i,s+1} = \begin{cases} K_{i,s} & \text{for } i < s, \\ E_s & \text{for } i = s, \end{cases}$$

$$y_{i,s+1} = \begin{cases} r_{i,s} & \text{for } i < s, \\ x_s & \text{for } i = s, \end{cases}$$

$G_{s+1} = (M_s \cup T_s \cup \{x_s\})^*$. □ End of Construction

Notice that at stage $s+1$ either $T_{s+1} \vdash y_{e,s+1} \leftrightarrow y_{e,s}$ as $\bar{P}_{j_s} \in T_{s+1} - T_s$ (our action is essentially irrelevant), or some P_{j_s} occurring in $y_{e,s}$ which might be forcing $T_{s+1} \vdash y_{e,s}$ is removed. By construction $\operatorname{card}(\bigcap_s F_{e,s}) \geqslant 2$, i.e., e requires attention at most $\operatorname{card}(E_e - 2)$ times, and it is easy to see the G we form is generated by

$$\{P_i \mid i \in A\} \cup \{\overline{P_j} \mid j \in B\} \cup \bigcup_{e\in\omega} \left\{\bigvee_{i\in F_e} P_i\right\}$$

where, for all e, $F_e \cap (A \cup B \cup \bigcup_{j\neq e} F_j) = \emptyset$. Consequently it follows that $G \neq {}^*T$ and result follows. □

This result yields

Corollary 3.6. *Suppose $\delta \neq 0$ is any nonzero r.e. degree. There exists an r.e. theory $T = (P_i, \overline{P_j} \mid i \in A, j \in B)^*$ such that A, B is a maximal pair and T is not weakly Martin–Pour-El and T has degree δ.*

Proof. By [1] or [5] an r.e. set C possesses a decomposition into an r.e. maximal pair if and only if C is simple. Let C be a simple set of degree δ which is not hypersimple and let A, B be the resulting decomposition. By 3.5, $A \cup B$ ought to be hypersimple if $T = (P_i, \overline{P_j} \mid i \in A, j \in B)^*$ were weakly Martin–Pour-El. □

Theorem 3.5 suggests that if T is Martin–Pour-El, perhaps $A \cup B$ is (say) maximal, or hyperhypersimple, etc. The following result is therefore interesting from two points of view: first it shows that this is not the case, second it is interesting since later we show there is an r.e. degree not bounding an r.e. degree containing a Martin–Pour-El theory.

Theorem 3.7. *There exists an r.e. Martin–Pour-El theory such that T has low degree (that is $T' \equiv_T \mathbf{0}'$).*

Remark 3.8. Consequently, by Martin's theorem [13] $A \cup B$ (for this theory) is not maximal, nor dense-, super-, hyperhyper-simple or any class containing only members of high r.e. degrees. This result may be proved by direct modification of the standard 'lowness' finite injury argument according to, say, Soare [20] (cf. [2] for details). Basically putting $C_s = A_s \cup B_s$ we define a restraint $r(e, s)$ such that $r(e, s) = 0$ if $\Phi_{e,s}(C_s; e)\uparrow$ and $r(e, s) = u(e, C_s, s, e)$ if $\Phi_{e,s}(C_s; e)\downarrow$. Now define $R(e, s) = \max\{e, r(i, s) \mid i \leqslant e\}$ and if R_e as before requires attention we put $\varepsilon_i \bar{B}_{i,s}$ into $T_{s+1} - T_s$ for $i > R(e, s)$. A simultaneous induction based on the usual argument shows that $\lim_s r(e, s) = r(e)$ exists, after all the R_j for $j < e$ are met, and then R_{j+1} requires attention at most $2^{2^{R(e)}}$ more times.

We may sharpen this result as follows using the Jockusch–Shore pseudo-jump technique [12].

Theorem 3.9. *Let n be given. There exists an r.e. Martin–Pour-El theory T such that $T \oplus W_n^T \equiv_T \mathbf{0}'$ where $T \oplus W_n^T$ denotes $\{2x \mid x \in T\} \cup \{2x + 1 \mid x \in W_n^T\}$.*

Remark 3.10. And so choosing an index n such that for all B, $W_n^B \equiv_T B'$, we may deduce the first result.

Proof. We again define $C_s = A_s \cup B_s$ and show $C \oplus W_e^C \equiv_T K$ where $K = \{\langle x, y\rangle \mid x \in W_y\}$. We retain the notation and terminology of Theorem 3.1 with the following changes. We must preserve computations showing $e \in W_n^C$. Define $r(e, s)$ to be the use function for computing $e \in W_{n,s}^{C_s}$ if $e \in W_{n,s}^{C_s}$ and $r(e, s) = 0$ otherwise. Now define $R(e, s) = \max\{e, r(i, s) \mid i \leqslant e\}$. Define $L(e, s, y) =$

$\{\overline{\varepsilon_i B_{i,s}} \mid \varepsilon_i B_{i,s}$ occurs in y and $i > R(e, s)\}$. We introduce markers on the natural numbers for convenience (in encoding K).

Construction
Stage 0. As in 3.1, and declare all $m \in \omega$ unmarked.
Stage $s+1$. Define $L = L(e, s, y)$ if $e \leqslant s$ and R_e requires attention via y, define $L = \emptyset$ if for no $e \leqslant s$, R_e requires attention. For $0 \leqslant m \leqslant s$ find the least unmarked number m, with $m \in K_s$, mark it, and define:

$$T_{s+1} = \begin{cases} T_s \cup L & \text{if no such } m \text{ exists,} \\ T_s \cup L \cup \{\bar{B}_{k,s}\} & \text{if } \bar{B}_{k,s} \in L, \text{ and } m \text{ exists,} \\ T_s \cup L \cup \{B_{k,s}\} & \text{otherwise,} \end{cases}$$

where $k = R(m, s) + 1$. Finally set $Q_{e,s+1} = Q_{e,s} \cup \{y\}$ if $L(e, s, y) \neq \emptyset$ and $Q_{i,s+1} = Q_{i,s}$ for $i \neq e$. If $L = \emptyset$, set $Q_{e,s+1} = Q_{e,s}$. □ End of Construction

As in 3.2 if y is acted on at stage $s+1$, then there exists x, a boolean combination of $B_{0,s}, \ldots, B_{R(e,s),s}$ such that $T_{s+1} \vdash y \leftrightarrow x$.

Lemma 3.11. $\lim_s R(e, s) = R(e)$ *exists, all the* R_e *are finite, all the* R_e *are met, and* $\lim_s B_{i,s} = B_i$ *exists.*

Proof. By induction, go to a stage t where for all $s \geqslant t$, $r(i, s) = r(i)$ for $i < e$, and all the P_j for $j < e$ have stopped requiring attention (and so $Q_{j,s} = Q_j$). Moreover, by induction we may suppose that any number $\leqslant e$, to be marked at any stage has been by stage t. Now if $e \notin W^{C_s}_{n,s}$ for any $s \geqslant t$, then $r(e, s) = 0$. If there exists $t' \geqslant t$ such that $e \in W^{C_{t'}}_{n,t'}$ (t' least) then $r(e, s) = r(e, t')$ for all $s > t'$ since we are protecting these computations. In any case $\lim_s R(e, s) = R(e)$ exists. Go to stage $t' \geqslant t$ where $\forall s > t\, (R(e, s) = R(e, t'))$. Now R_e can require attention at most $2^{2^{R(e,t')}}$ more times. □

Lemma 3.12. $T \oplus W^T_n \equiv_T K$.

Proof. As $\lim_s r(e, s) = r(e)$ exists, the limit lemma ensures that $W^T_n \leqslant_T K$: using the K-oracle find a stage t where $r(i, s)$ for $i \leqslant e$ has reached its final value, now see if $r(i, s) = 0$.

Conversely suppose we are given a C- and a W^C_n-oracle. Let $m \in \omega$, and suppose we can compute a stage t where, for all $e < m$, for all $s > t\, (r(e, s) = r(e, t))$. Via the C-oracle we may then compute a stage t' where for all $e < m$ and all $j < R(e, s) + 1$, $B_{i,s} = B_i$, for all $s > t'$. Now if the current computation concerning ($m \in W^{C_{t'}}_n$?) is later destroyed, it is because $W^{C_t}_{n,t}$ later changes through its current use function. We may find a stage v via this oracle and the C-oracle where the computations computing $m \in W^{C_v}_{e,v}$? are permanent, and via the

C-oracle, where for all $i \leqslant R(e, v)+1$, for all $s \geqslant v$, $B_{i,s} = B_{i,v}$. Now $R(e, v) = R(e)$. And moreover, $m \in K \leftrightarrow m \in K_v$ since, if $\exists s > v\ (m \in K_s - K_v)$, then the construction endures that $\exists s_1 \geqslant s\ (B_{R(e)+1,s_1} \in C_{s_1+1} - C_s)$. □

In view of the results of Jockusch and Shore [10] this gives

Theorem 3.12. *Let $n \in \omega$. There exist r.e. Martin–Pour-El theories T_1, T_2 with* $\deg(T_1)$ *in* $\text{low}_{n+1} - \text{low}_n$ *and* $\deg(T_2)$ *in* $\text{high}_{n+1} - \text{high}_n$.

4. Degrees

The main result of this section is to produce an r.e. nonzero degree which bounds no r.e. Martin–Pour-El theory. This is somewhat surprising in view of the fact that we show they exist in low and high r.e. degrees, and also because of the following theorem.

Theorem 4.1. *Let D be any r.e. nonrecursive set. There exists a weakly Martin–Pour-El theory T with $T \equiv_T D$.*

Proof. We build $T = \bigcup_s T_s$ and retain the notation of Theorem 3.1. Let f be a 1–1 recursive function enumerating D. To ensure $T \geqslant_T D$ we code, that is, we ensure $\varepsilon B_{f(s),s} \in T_{s+1} - T_s$ some ε. To ensure $T \leqslant_T D$ we permit on the index of the complement. We introduce therefore,

$$n(e, s, y) = \begin{cases} \text{the } \textit{least } m > e \text{ such that } \varepsilon_m B^s_m \text{ occurs in } y, & \text{if one exists,} \\ -1, & \text{otherwise.} \end{cases}$$

Our requirements are

R_e: $W_e \supset T \rightarrow W_e =^* T$,

N_e: $\lim_s B_{e,s} = B_e$ exists.

We say R_e *requires attention at stage* $s+1$ via y if y is least for the least e such that $y \in W_{e,s+1}$ and $y \notin (T_s, Q_{e,s})^*$ and $n(e, s, y) \geqslant f(s)$.

Construction

Stage $s+1$. If no R_e requires attention, set $T_{s+1} = (T_s, B_{f(s),s})^*$. If R_e requires attention via y define

$$T_{s+1} = \begin{cases} (T_s, L(e, s, y))^* & \text{if } \bar{B}_{f(s),s} \in L(e, s, y), \\ (T_s, L(e, s, y), B_{f(s),s})^*, & \text{otherwise.} \end{cases}$$

Define $Q_{e,s+1} = Q_{e,s} \cup \{y\}$. □ End of Construction

As before, if y is acted on at stage $s+1$, there exists z, a boolean combination of $B_{0,s}, \ldots, B_{e,s}$ such that $T_{s+1} \vdash y \leftrightarrow z$. Consequently, $\lim_s B_{e,s} = B_e$ exists, since R_e can receive attention at most finitely often. It remains to show that all the R_e are met and that $T \equiv_T D$. This follows from the following lemma which was proved jointly with Jeff Remmel. (In the original version of this paper, there was an error in the proof of this lemma.)

Lemma 4.2 (with Remmel). *Suppose* $N = (P_i, \bar{P}_j : i \in A,\ j \in B)^*$ *is any well generated consistent theory, and* $\{B_0, B_1, \ldots\}$ *list in order* $\{P_i : i \notin A \cup B\}$. *Suppose* W *is any consistent theory with* $W \supset N$ *and* $W \neq^* N$. *Then for any* n *there exists* $x \in W_e$ *with* $x = \bigvee_{i \in F} \varepsilon_i B_i$ *and* $m \in F$ *implies* $m > n$.

Remark. An equivalent form of this lemma is: Let n be given and suppose W is any consistent theory with $W \neq^* \{1\}$, then there exists $x \in W$ with $x = \bigvee_{i \in F} \varepsilon_i P_i$ and for all $i \in F$, $i > n$.

Proof of 4.2. Suppose the hypotheses are satisfied. Then for each $z \in W - N$ we have

$$N \vdash z \leftrightarrow \bigvee_{i \leq n} \varepsilon_i B_i \vee \bigvee_{i > n} \varepsilon_i B_i = x \vee Y \quad \text{with } x \neq 0.$$

There are at most 2^{2^n} choices for x, thus $x \in \{x_1, \ldots, x_m\}$ say. Consequently we have that W is a consistent extension of N such that for all $z \in W - N$ we have

$$N \vdash z \leftrightarrow x_i \vee y \quad \text{with } x_i \neq 0,$$

where $x_i = \bigvee_{j \leq n} \varepsilon_j B_j$ and $y = \bigvee_{j > n} \varepsilon_j B_j$. We need to show $W =^* N$. We prove this by induction on m (but independent of n). Without loss, we assume that for all ε, i, $\varepsilon B_i \notin W$ (otherwise add εB_i to N, etc). Now if $m = 1$, then $W \subseteq (N, x_1)^*$. Assume the result for $m \leq k$, and consider $m = k+1$. Define i to be *bad* if there exist j, k with $x_j = B_i$ and $x_k = \bar{B}_i$. There are two cases.

Case 1: $\exists i$ *($i \leq n$ & i is not bad)*. Without loss suppose $i = 1$ and B_1 occurs in x_j implies that for some $t = t(j)$, εB_t occurs in x_j with $t \leq n$ and $t \neq 1$.

Subcase (i): $\exists p$ *($\bar{B}_1$ occurs in x_p)*. In this subcase, define $W' = (W, \bar{B}_1)^*$. Notice that $0 \notin (W, \bar{B}_1)^*$, for if $0 \in (W, B_1)^*$, then $0 = \bar{B}_1 \wedge y$ for some $y \in W$, and so $B_1 = B_1 \vee y$ implying $B_1 \in W$, contradiction. Define $N' = (N, \bar{B}_1)^*$. Let $\{x_1, \ldots, x_t\}$ list those x_i not containing $\bar{B}_1$ (notice $t \leq k$). Now let x_i' be the result of deleting all occurrences of B_1 from x_i. Notice that *as* 1 *is not bad*, for all i with $1 \leq i \leq t$, $x_i' \neq 0$. Now we see that W' is a consistent extension of N' such that for all $z \in W' - N'$,

$$N' \vdash z \leftrightarrow x_i' \vee y \quad \text{with } x_i' \neq 0.$$

Now we apply the induction hypothesis to W', N' since $t \leq k$.

Subcase (ii): $\forall p$ ($\bar{B}_1$ does not occur in x_p). This subcase is easy. Define

$W' = (W, B_1)^*$ and $N' = (N, B_1)$, and apply the induction hypothesis to W', N' and $\{x_2, \ldots, x_t\}$ where $x_2, \ldots, x_t$ list those x_i not containing occurrences of B_1.

Case 2: All $i \leqslant n$ are bad. In particular, for (w.l.o.g.) $i = 1$ we have $x_1 = B_1$ and $x_2 = \bar{B}_1$. Now fix y where

$$y = B_1 \vee \bigvee_{i>n} \varepsilon_i B_i = B_1 \vee y_1.$$

Let $q \in W - N$ be such that 1 *is bad for q, namely, if $\varepsilon_i B_i$ occurs in q and $i \leqslant n$, then $i = 1$ and $\varepsilon_i B_i = \bar{B}_1$.* Then

$$N \vdash q \leftrightarrow \bar{B}_1 \vee \bigvee_{i>n} \varepsilon_i B_i = \bar{B}_1 \vee q_1.$$

It follows that

$$\begin{aligned} y \wedge q &= (B_1 \vee y_1) \wedge (\bar{B}_1 \vee q_1) \\ &= (B_1 \wedge \bar{B}_1) \vee (B_1 \wedge q_1) \vee (\bar{B}_1 \wedge y_1) \vee (y_1 \wedge q_1) \\ &= (B_1 \wedge q_1) \vee (\bar{B}_1 \wedge y_1) \vee (y_1 \wedge q_1) \\ &= (y_1 \vee q_1) \wedge [\cdots], \end{aligned}$$

implying $y_1 \vee q_1 \in W$. It follows that $y_1 \vee q_1 = 1$ since if $y_1 \vee q_1 \neq 1$, we see $y_1 \vee q_1 = \bigvee_{i>n} \varepsilon_i B_i$ by construction, contradicting the hypotheses of the lemma. Thus for all $q \in W - N$, if 1 is bad for q, then $q = \bar{B}_1 \vee q_1$ and $y_1 \vee q_1 = 1$. Thus for all q, if 1 is bad for q, then for some $i > n$ with $\varepsilon_i B_i$ in y_1, we have $\overline{\varepsilon_i B_i}$ occurring in q_1. Let

$$W' = (W, B_1, \overline{\varepsilon_i B_i} : \varepsilon_i B_i \text{ occurs in } y_1 \;\&\; i \neq 1)^*,$$

and

$$N' = (N, \overline{\varepsilon_i B_i}, B_1 : \varepsilon_i B_i \text{ occurs in } y_1 \;\&\; i \neq 1)^*.$$

We claim W' is consistent. Suppose $0 \in W'$. Then $0 = B_1 \wedge \bigwedge_{i>n} \overline{\varepsilon_i B_i} \wedge x$ for some $x \in W$. This means $\bar{B}_1 \vee \bigvee_{i>n} \varepsilon_i B_i = \bar{B}_1 \vee \bigvee_{i>n} \varepsilon_i B_i \vee x \in W$. But recall that $y \in W$ where $y = B_1 \vee y_1 = B_1 \vee \bigvee_{i>n} \varepsilon_i B_i$. It follows that $\bigvee_{i>n} \varepsilon_i B_i \in W$, a contradiction. Thus $0 \notin W'$. Let $\{x_2, \ldots, x_t\}$ denote those x_i not containing B_1 nor $\overline{\varepsilon_i B_i}$ as above, and let x_i' denote the result of deleting $\bar{B}_1$ from x_i. Then W' is a consistent extension of N' such that for all $z \in W' - N'$ we have $N' \vdash z \leftrightarrow x_i' \vee p$ where $x_i' \neq 0$ with $x_i \in \{x_2, \ldots, x_t\}$. (Notice that $x_i' \neq 0$ since if x_i contains $\bar{B}_1$ and $x_i \in \{x_2, \ldots, x_t\}$, then 1 is not bad for x_i.) Applying the induction hypothesis to W', N' and $\{x_2, \ldots, x_t\}$ will conclude the proof of 4.2. □

In view of this result, if we wish to produce an r.e. degree free of Martin–Pour-El theories, we must at least have a technique which produces a weakly Martin–Pour-El theory which is not Martin–Pour-El in some r.e. degree. As the next result actually improves on this, and gives the basic strategy for one negative requirement in the subsequent theorem, we give the result in some detail.

Theorem 4.3. *Suppose* $\boldsymbol{\delta}$ *is any nonzero r.e. degree. Then* δ *contains an r.e. weakly Martin–Pour-El theory* T *which is not Martin–Pour-El.*

Proof. We suppress the degree requirements (which are not serious obstacles and are met by permitting and coding) until the end of the theorem, as we feel they will interfere with the exposition of the basic strategy. To ensure T is weakly Martin–Pour-El we meet

R_e: $W_e \supset T$ implies $W_e =^* T$.

To ensure T is not Martin–Pour-El, we must produce an r.e. theory M with $0 \neq M$, $M \supset T$ and M not principal over T. To this end, we define an ordering $\ll$ of the elements in $\bigvee \varepsilon_i P_i$ form via

$$(1 \neq) \bigvee_{i \in I} \varepsilon_i P_i \ll \bigvee_{j \in J} \varepsilon_i P_i \, (\neq 1)$$

if for all $i \in I$ there exists $j \in J$ such that $\varepsilon_i P_i = \varepsilon_j P_j$. Notice $y \ll x$ if and only if $\vdash y \rightarrow x$. The idea is to produce an infinite collection $\{z(e) \mid e \in \omega\}$ of elements of the form $\bigvee \varepsilon_i P_i$ such that $M = (T, z(e) \mid e \in \omega)^* \nvdash 0$ and for all k, $z(k) \notin (T, z(e) \mid e \neq k)^*$, and moreover, for all $y \ll z(e)$, $y \neq z(e)$, $y \notin M$. We must be careful with the interaction of the construction of the $z(e)$'s and the satisfaction of the R_e. We construct $z(e) = \lim_s z(e, s)$. We must be careful that although we can satisfy the R_j we cannot too seriously injure work we've done in constructing $z(e, s)$, say. The point is that suppose we choose $z(e) = P_k$, then we cannot ever add $\overline{P_k}$ to T for then $M \vdash 0$. This has a serious effect on the R_j for $j < k$, i.e., higher priorities are affected by lower ones. (This type of action in fact is trying to stop T from being weakly Martin–Pour-El.)

Our solution is to *link* the $z(e)$'s together, so that at the end of the construction, the $z(e)$ will appear as

$$\begin{aligned}
z(0) &= g_0 \vee g_1 \vee g_2 \, (= d_0^0 \vee d_1^0 \vee d_2^0),\\
z(1) &= g_0 \vee g_1 \vee g_3 \vee g_4 \, (= d_0^1 \vee d_1^1 \vee d_2^1 \vee d_3^1),\\
z(2) &= g_0 \vee g_1 \vee g_3 \vee g_5 \vee g_6 \, (= d_0^2 \vee d_1^2 \vee d_2^2 \vee d_3^2 \vee d_4^2),\\
&\dots \qquad\qquad \dots
\end{aligned}$$

where the $z(i)$ are 'linked' by $(g_0, g_1, g_3, \dots)$, where each g_i is a P_j for some j $(g_i \neq g_k)$. This idea allows us to never add 0 to M through our efforts to satisfy the R_e. We must satisfy

N_e: There exists $z(e)$ as described previously, where $\lim_s z(e, s) = z(e)$.

Initially all the $z(e)$ are undefined. At each stage s we ensure that $z(0, s), \dots, z(t, s)$ are defined, where $t \rightarrow \infty$ as $s \rightarrow \infty$. The ranking is

$N_0, R_0, N_1, R_1, \dots$

Notice that N_0 is never injured provided our protection is satisfactory. We say that R_e *requires attention at stage* $s+1$ if

(i) $d_{e+2}^{e+1,s}$ is defined,

(ii) $0 \notin (W_{e,s}, T_s)^*$,

(iii) there exists $y \in W_{e,s}$ such that if $g(y) = \min\{\varepsilon_i P_i \mid \varepsilon_i P_i$ occurs in $y\}$, then $g(y) > d_{e+2}^{e+1,s}$,

(iv) for some set J, $y = \bigvee_{j \in J} \varepsilon_i B_{j,s}$,

(v) e is least, with respect to (i), (ii), (iii) & (iv).

We say that R_e requires attention *via* y, in this case. (We presuppose an ordering of the P_i's with P_i, $\bar{P}_i < \min\{P_{i+1}, \overline{P_{i+1}}\}$.)

For technical reasons, we employ a sequence $t_{0,s} < t_{1,s} < \cdots$ of elements 'yet to be involved', formally $t_{i,s}$ is the $i+1$-th P_i with P_i, $\bar{P}_i \notin T_s$ and $P_i \neq d_j^{k,s}$ for any j, k, or $t \leq s$ (where defined). We say N_e *requires attention* if $z(e, s)$ is currently undefined.

Construction

Stage 0. Set $T_0 = \emptyset$, $d_0^{0,0} = P_0$, $d_1^{0,0} = P_1$, and $d_2^{0,0} = P_2$, and so $z(0, 0) = P_0 \vee P_1 \vee P_2$. Now define $t_{j,0} = P_{j+3}$ for all $i \in \omega$.

Stage $s+1$. If no N_e nor R_e for $e \leq s$ requires attention we may suppose N_{s+1} requires attention. Thus either N_e requires attention for $e \leq s+1$ or R_f requires attention for $f \leq s$. If N_e requires attention (with $e = f+1$, as $e \neq 0$) define

$$z(i, s+1) = z(0, s) \quad \text{for } i \leq f$$

and

$$z(e, s+1) = d_0^{f,s} \vee \cdots \vee d_{f+1}^{f,s} \vee t_{0,s} \vee t_{1,s}$$

and so $d_i^{e,s+1} = d_i^{f,s}$ for $i \leq e$, $d_{e+1}^{e,s+1} = t_{0,s}$ and $d_{e+2}^{e,s+1} = t_{1,s}$. Now set $t_{i,s} = t_{i+2,s}$ for all $i \in \omega$. Declare as undefined all the $z(i, s+1)$ for $i > e$. If R_e requires attention via y, define

$$L = L(e, s, y) = \{\overline{\varepsilon_i P_i} \mid \varepsilon_i P_i \text{ occurs in } y\}.$$

(Notice that as $y = \bigvee \varepsilon_i B_i^s$, $0 \notin (T_s, L)^*$.) Define $T'_{s+1} = (T_s, L)^*$.

We now perform a *recovery step* which forces all the $z(j, s)$ for $j > e$ into T_{s+1}. Our strategy has protected $d_{e+2}^{e+1,s}$ which, by induction occurs in all the $z(j, s)$ for $j > e$ (i.e., $d_{e+2}^{j,s} = d_{e+2}^{e+1,s}$). So the recovery step is to set

$$T_{s+1} = (T'_{s+1}, d_{e+2}^{e+1,s})^*.$$

Notice (again, by induction) this has no effect on the $z(i, s)$ for $i \leq e$. We may now generate a $t_{i,s}$ list by deleting from the current $t_{i,s}$ list all the $t_{i,s} = \varepsilon P_j$ such that $\varepsilon P_j \in T_{s+1} - T_s$, giving, say $t_{0,s} < \cdots < t_{e,s} < \cdots$. Now finally declare all the $z(j, s)$ for $j > e$ undefined, and go to stage $s+2$. □ End of Construction

The reader may check that once we reach a stage s where for all $s \geq t$, $z(j, s) = z(j, t) = z(j)$, $j \leq e$, then all the P_j for $j < e$ are met. Now wait till a stage

$s > t$ occurs where $z(e+1, s)$ is defined. Now the action (waiting for $y \in W_{e,s}$ and putting $\bar{y} \in T_{s+1}$) of the R_e ensures that R_e requires attention at most once, that is, $z(e+1, s)$ once defined can become undefined at most once more. Therefore all the N_e are met for M will be, by induction, as described in the original specifications. Finally all the R_e are met by Lemma 4.2: Once a stage t occurs such that $\forall s > t\,(z(e+1, s)$ is defined), $z(e+1, t) = z(e+1)$. It follows that $\exists z \in W_e\,(z = \bigvee \varepsilon_i B_i$ such that $\varepsilon_i B_i > d_{e+2}^{e+1,t})$. However letting $F = \{i \mid \varepsilon_i B_i \leq d_{e+2}^{e+1,t}\}$, then F is finite, and so every element of W_e not in T is equivalent to one of the form

$$y = \bigvee_{i \in F'} \varepsilon_i B_i \vee \bigvee_{i \notin F} \varepsilon_i B_i,$$

i.e., where $F' \subset F$ and $F' \neq \emptyset$, and under these conditions, 4.2 says that $W_e =^* T$.

We now conclude the proof by giving some remarks concerning how one may introduce the degree-theoretic restrictions. As we remarked earlier, we shall achieve this using standard permitting and coding. We feel that it is completely clear that, if one permits on indices, the result will still hold, viz: put $\varepsilon_i B_{i,s}$ only if (i) $B_{i,s} > d_j^{k,s}$ as before and (ii) all the i for $\varepsilon_i B_{i,s}$ in x are $\geq f(s)$ where f is some desired recursive function.

However, coding some set $f(\omega)$ into T is a little less straightforward, in the sense that it will really injure our N_e strategy. We actually encode $\varepsilon B^s_{2f(s)+5,s}$ or $\varepsilon B^s_{2f(s)+6,s}$ into $T_{s+1} - T_s$, and perform a recovery step after each encoding. Thus we define

$$z(0, 0) = P_0 \vee \cdots \vee P_4 = B_0 \vee \cdots \vee B_4$$

so that

$$z(1, s) = B_0 \vee \cdots \vee B_3 \vee B_{5,s} \vee B_{6,s}.$$

Now the requirement R_1 is the only positive one to affect $z(1, s)$, and the worst it can do is add $\varepsilon B_{6,s}$ to $T_{s+1} - T_s$, and we then recover by adding $B_{5,s}$ to T_{s+1} so that all the $z(i, s)$ for $i > 1$ are annihilated at this stage. Now thereafter one can check that for some $s' > s$

$$z(1, s) = B_0 \vee \cdots \vee B_3 \vee b_{5,s'} \vee B_{6,s'}.$$

Now at worst the coding requirement asks us to add one of $\varepsilon B_{5,s'}$ or $\varepsilon B_{6,s'}$ to $T_{s+1} - T_s$. By selecting this to be $\varepsilon B_{6,s'}$ we may proceed as before, in the sense that this action is precisely as if it were attacked by an R_1. Similarly one can check (by induction) that if $z(e, s)$ is defined, then

$$z(e, s) = B_0 \vee \cdots \vee B_3 \vee B_{5,s'} \vee B_{7,s} \vee \cdots \vee B_{2e+3,s}$$

and our above comments apply to $z(e, s)$ as they did to $z(1, s)$. (Notice here the recovery step is required for the above configuration for $z(e, s)$. Without this, as in the next construction, it is unclear how one may 'force degrees upward'.) The reader may check that the above remarks indeed give the desired result. □

We are now in a position to show:

Theorem 4.4. *If $\boldsymbol{\delta} \neq \mathbf{0}$ is any r.e. degree, then there exists an r.e. degree $\boldsymbol{\delta}_1 \neq \mathbf{0}$ below $\boldsymbol{\delta}$ which bounds no r.e. Martin–Pour-El theory.*

Proof. We construct an r.e. set $D = \bigcup_s D_s$ in stages. For technical convenience, we shall delete the requirement that we construct D below a given r.e. degree. This is obtained in the obvious way via standard permitting, and really in no way interferes with the construction. We first must ensure that D is nonrecursive. Recalling that w_e is the e-th r.e. set, we satisfy

R_e: $\bar{D} \neq w_e$.

Our negative requirements will be similar to those of the previous theorem in some sense.

N_e: If $\Phi_e(D) = A_e \cup B_e$ and $\text{card}(\omega - (A_e \cup B_e)) = \infty$, then $(P_i, \overline{P_j} \mid i \in A_e, j \in B_e)^*$ is not Martin–Pour-El.

(Where $\langle \Phi_e, A_e, B_e \rangle$ is an enumeration of the triples of oracles, and disjoint r.e. sets.)

We satisfy these as

N_e': If $\Phi_e(D) = A_e \cup B_e$ and $\text{card}(\omega - (A_e \cup B_e)) = \infty$, then there exists a collection $z(e) = \lim_s z(e, s)$ which have the same properties relative to D as did the $z(e)$ in Theorem 4.3.

Let us first consider the interaction of one N_e' with the satisfaction of the R_e. In Theorem 4.3 we built the $z(e)$ as $z(0, s), \ldots, z(k, s)$ where $k \to \infty$. Each $z(i, s)$ was of the form $v(i, s) \vee m \vee n$ where $z(i-1, s)$ was of the form $v(i, s) \vee g$ and $g \neq m$. We ensured that we did not allow the $v(i, s)$ to be forced into T due to the action of any R_j for $j > e$. In some sense we could use the same strategy here since once the length of agreement between $\Phi_{e,s}(D) = A_{e,s} \cup B_{e,s}$ reaches a certain length such that a number of elements are *excluded* from $A_e \cup B_e$, we can keep those elements out of $A_e \cup B_e$ forever by restraining D on the use function associated with the computation. That is, we wait until, say,

$$\Phi_{e,s}(D, x) = (A_{e,s} \cup B_{e,s})[x] \quad \text{for all } x \leq l(e, s)$$

(where $l(e, s)$ is the current length of agreement, and the computation has used u), such that for $P_1 \neq P_2 \neq P_3$, $\{P_1, P_2, P_3\} \cap (A_{e,s} \cup B_{e,s}) = \emptyset$ and $P_i < l(e, s)$. We now define $z(0, s) = P_1 \vee P_2 \vee P_3$ and refrain from enumerating any new elements into D below u. Later, we might get new elements P_4, P_5 not in $(A_{e,s'} \cup B_{e,s'})$ when the length of agreement l' rises so that $l' > \max\{P_1, \ldots, P_5\}$. We can then define $z(1, s') = P_1 \vee P_2 \vee P_4 \vee P_5$ and restrain on the use of $\Phi_{e,s'}(D_{s'}, x) = (A_{e,s'} \cup B_{e,s'})[x]$ for $x \leq l'$. In this way we might permanently define our sequence $z(0), z(1), \ldots$.

The first problem is that we must spread out the defining of the various $z(e)$'s so they may interact with the R_j. This is solved essentially by the technique of 4.3, we replace N'_e by

$N_{\langle e,k\rangle}$: *If* $\Phi_e(D) = A_e \cup B_e$ and $\text{card}(\omega - (A_e \cup B_e)) = \infty$, then for all $i \leqslant k$,
4 $\lim_s z(\langle e, i\rangle, s) = z(\langle e, i\rangle)$ exists with the desired properties.

We assume, given e, that $\langle e, \cdot\rangle$ is monotone increasing in the second variable for a fixed first variable.

The second problem is that we can no longer perform the recovery step of the previous result. A moment's thought reveals however that this is not really necessary, provided we retain any protection we previously imposed, in the sense that this will give us a larger list, say

$$\{z(e)\} \cup \{q(e)\}.$$

The third problem comes about because we are dealing with many N_e's. This is more serious since between $N_{\langle e,k\rangle}$ and $N_{\langle e,k+2\rangle}$ there may be many R_j each of which might injure our current candidate for $z(e, s)$. Indeed it is clear that $N_{\langle e,0\rangle}$ might be *fatally* injured by the R_j for $j < \langle e, n\rangle$. This is overcome by starting our definition of the $z(\langle e, 0\rangle, s)$ over again, and a finite injury argument eventually reveals that $z(\langle e, 0\rangle)$ will become permanently defined. The real obstacle is that between $N_{\langle e,k\rangle}$ and $N_{\langle e,k+2\rangle}$ there are many R_j, and yet we cannot keep redefining $z(\langle e, i\rangle, s)$ for $i \leqslant k$, to keep all the linkages intact. The idea is now to take (say) $z(\langle e, 0\rangle, s) = P_1 \vee P_2 \vee \cdots \vee P_t$ where $t > (\langle e, 1\rangle - \langle e, 0\rangle) + 4$. In this way, whenever $N_{\langle e,1\rangle}$ is injured, we shall be able to pick, at some future stage, another set $\{P'_{i_1}, \ldots, P'_{i_j}\}$ and thus after no injuries,

$$z(\langle e, 1\rangle, s) = P_1 \vee P_2 \vee \cdots \vee P_{t-1} \vee P_{i_1} \vee \cdots \vee P_{i_j}.$$

But,

$$z(\langle e, 1\rangle, s) = P_1 \vee P_2 \vee \cdots \vee P_{t-2} \vee P'_{i_1} \vee \cdots \vee P'_{i_j}$$

after the first injury, etc. This idea will show that after $N_{\langle e,1\rangle}$ is maximally injured, i.e. at most $t-4$ times, then $z(\langle e, 1\rangle, s\rangle = P_1 \vee P_2 \vee P_3 \vee P'_{i_1} \vee \cdots \vee P'_{i_j}$ and now no element $\ll z(\langle e, 1\rangle, s)$ may enter $A_e \cup B_e$.

For technical convenience for each e it will be useful to generate the set $\{B^e_{i,s} \mid i \in \omega\} = M_s$ where if $B^e_{j,s} = P_i$ iff $i \notin A_{e,s} \cup B_{e,s}$. Define a function $g(e, k) = (\langle e, k+1\rangle - \langle e, k\rangle) + 4$. We define a list $a_{i,s} = \{x \in \omega \mid x \notin D_s\}$. We actually make D simple and we ensure $\lim_s a_{i,s} = a_i$ exists. At each stage we generate certain restraints $r(e, s)$ and define $R(e, s) = \max_{i \leqslant e}\{r(i, s), a_{i,s}\}$. We say R_e *requires attention via* x if x is least such that $x \in w_{e,s}$ and $x > R(e, s)$ and $w_{e,s} \cap D_s = \emptyset$ (where e is least).

In the construction to follow we shall employ certain technical devices, hopefully to simplify notation

$$v(i, s) = \text{the } \textit{critical part of } z(i, s),$$

$$Y(i, s) = \text{the set of } \textit{rejected } z(i, s).$$

If $z(i, s)$ is defined, then it will always be the case that $v(i, s)$ is defined. We define $v(-1, s)=0$. We say N_j is *injured* if R_i for $i<j$ requires attention. If $N_j=N_{\langle e,k\rangle}$ and R_i requires attention for $i<\langle e, 0\rangle$ we say N_j is *fatally injured.* Finally we say $N_{\langle e,k\rangle}$ *requires attention at stage* $s+1$ if $z(\langle e, k\rangle, s)$ is undefined, $N_{\langle e,k-1\rangle}$ is unsatisfied, and there exist a set J of $B_{i,s}$ such that $J=\{B^e_{j0,s}, \ldots, B^e_{j_g(e,k),s}\}$ (i.e., $\text{card}(J)>g(e, k)$) and if $i_s=\max\{i \mid P_i=B^e_{j_k}$ for some $j_k\in J\}$, then the current length of agreement $l(e, s)$ of the computation

$$\Phi_{e,s}(D_s; x)=(A_{e,s}\cup B_{e,s})(x)$$

is greater than i_s (that is $\forall y\leqslant i_s+1$ $(\Phi_{e,s}(D_s; y)=(A_{e,s}\cup B_{e,s})(y))$. We say that $N_{\langle e,k\rangle}$ requires attention via $J=J(\langle e, k\rangle, s)$.

Finally for all $i\geqslant 1$, $v(i, s)$ will be of the form $P_{i_1}\vee P_{i_2}\vee\cdots\vee P_{i_m}$, say, with $P_{i_1}<\cdots<P_{i_m}$. We ensure that $\text{card}(i_1, \ldots, i_m)>3$ at any stage, and define the *next* $v(i, s)$ to be the element $n(i, s)$ where $v(i, s)=n(i, s)\vee P_{i_m}$, i.e., $n(i, s)=P_{i_1}\vee\cdots\vee P_{i_{m-1}}$. The element $P(i, s)=P_{i_m}$ will be called the *difference between* $v(i, s)$ and $n(i, s)$.

Construction

Stage 0. Set $D_0=\emptyset$ and $r(i, 0)=0$ for all $i\in\omega$, and finally define $a_{i,0}=0$ for all $i\in\omega$.

Stage s + 1. If no R_e, or N_e for $e\leqslant s$ requires attention, go to stage $s+2$. If R_e requires attention via x set $D_{s+1}=D_s\cup\{x\}$. Now for all $i>e$, declare $z(i, s+1)$ as being undefined. For all $i>e$, if N_i is fatally injured declare $Y(i, s)=\emptyset$ and $v(i, s)$ as undefined if $i\neq\langle k, 0\rangle$ and $v(\langle k, 0\rangle, s)=0$ all k, and finally declare $n(i, s)$ as undefined. If N_i is not fatally injured and $v(i, s)$ is currently defined, there are two cases.

Case 1. If $z(i, s)$ is currently defined, define $v(i, s+1)=n(i, s)$ and set $Y(i, s+1)=Y(i, s)\cup\{z(i, s)\}$.

Case 2. If $z(i, s)$ is undefined, define $v(i, s+1)=v(i, s)$ and set $Y(i, s+1)=Y(i, s)$.

Find the least i such that $a_{i,s}=x$ and set $a_{j,s+1}=a_{j,s}$ for $j<i$ and $a_{j,s+1}=a_{j+1,s}$ for $j\geqslant i$. Now go to stage $s+2$, declaring all injured N_i as unsatisfied at stage $s+1$, and maintaining all current restraints.

If $N_{\langle e,k\rangle}$ requires attention, and again is of highest priority, and it requires attention via $J=J(\langle e, k\rangle, s)$, we define

$$z(\langle e, k\rangle, s)=v(i, s)\vee P_{i_0}\vee\cdots\vee P_{i_{g(e,k)}}$$

where $\{P_{i_0}, \ldots, P_{i_{g(e,k)}}\}=J=\{B^e_{j0,s}, \ldots, B^e_{j_{g(e,k)},s}\}$. Now define $v(\langle e, k+1\rangle, s)$, to be

$$v(\langle e, k+1\rangle, s)=v(i, s)\vee P_{i_0}\vee\cdots\vee P_{i_{g(e,k)-1}}$$

and so

$$n(\langle e, k+1\rangle, s)=v(i, s)\vee P_{i_0}\vee\cdots\vee P_{i_{g(e,k)-2}}.$$

Declare $N_{\langle e,k\rangle}$ as currently satisfied. Notice that this aspect of the construction ensures that $v(i, s)$ is defined *before* N_i can require attention. (Recall here that $v(\langle e, 0\rangle, s) = 0$ *always*.) In this way we always come up with a linked system.

Finally raise the restraint $r(\langle e, k\rangle, s)$ to be

$$r(\langle e, k\rangle, s+1) = 1 + \max\{R(\langle e, k\rangle, s), u\}$$

where $R(\langle e, k\rangle, s) = \max\{e, r(i, s) \mid i \leq \langle e, k\rangle\}$ and u is the use function of the computation

$$\Phi_{e,s}(D_s; x) = (A_{e,s} \cup B_{e,s})(x) \quad \text{through length } l(e, s).$$

Now for all $i < \langle e, k\rangle$ define $r(i, s+1) = r(i, s)$ and for all $i > \langle e, k\rangle$ define $r(i, s+1) = \max\{r(\langle e, k\rangle, s+1), R(i, s)\}$ □ End of Construction

The reader may easily establish by induction that for all $e \in \omega$, for all k,

(i) If $\Phi_e(D) = A_e \cup B_e$ and $\text{card}(\omega - (A_e \cup B_e)) = \infty$, then

(a) $z(\langle e, 0\rangle, s)$ becomes defined at some stage s after which it is not fatally injured (so that all the R_j for $j < \langle e, 0\rangle$ have stopped requiring attention),

(b) once $z(\langle e, i\rangle, s)$ is defined, and never injured thereafter, $v(\langle e, i+1\rangle, s)$ is henceforth defined, and it can be injured at most $g(e, 4)$ times, and so if $\Phi_e(D) = A_e \cup B_e$ and $\text{card}(\omega - (A_e \cup B_e)) = \infty$, there exists a stage t where $z(\langle e, i+1\rangle, t)$ becomes permanently defined, $z(\langle e, i+1\rangle, t) = z(\langle e, i+1\rangle, s)$ for all $s > t$, and $\text{card}(v(e, i+1), t) \geq \text{card}(v(e, i)) + 2$ (with the obvious meaning).

(ii) If $\Phi_e(D) \neq A_e \cup B_e$ or $\text{card}(\omega - (A_e \cup B_e)) < \infty$, they cease to matter at some stage.

(iii) $\lim_s R(e, s) = R(e)$ exists and is finite.

(iv) All the R_j are met (and require attention at most once).

(v) $\lim_s a_{i,s} = a_i$ exists.

Finally notice that once $z(\langle e, 0\rangle, s)$ becomes permanently defined, say at stage t, we may consider $Y = \bigcup_{s>t} Y(i, s)$. This is r.e., and for all $s > t$, $Y(i, s) \neq \emptyset$. We must note that if $z(\langle e, i\rangle) = \lim_s z(\langle e, i\rangle, s)$, and if $\Phi_e(D) = A_e \cup B_e$ and if $\text{card}(\omega - (A_e \cup B_e)) = \infty$, then $(A_e \cup B_e \cup Y \cup \{z(\langle e, i\rangle) \mid i \in \omega\})^*$ is not principal over $(A_e \cup B_e)^*$ since one an show by induction that for all $q \ll z(e, i)$, $q \ll z(e, i) \notin (A_e, B_e, Y, z(\langle e, j\rangle) \mid j \neq i)^*$, and from this the result follows. □

In view of the above, the classification of the degrees containing Martin–Pour-El theories becomes very interesting, especially in view of the fact that there exist such theories in low degrees. The next result shows how to combine the basic construction with an infinite injury argument, and also extends our lattice development. We have:

Theorem 4.5. *Let $\emptyset <_T D <_T \emptyset'$. There exists an r.e. Martin–Pour-El theory $T = (P_i, \overline{P_j} \mid i \in A, j \in B)^*$ with $A \cup B$ a maximal set and $D \not\leq_T T$.*

Proof. Again, we build $T=\bigcup_s T_s$, and $C=\bigcup_s C_s$ where $C_s=A_s\cup B_s$. Our requirements are that C is maximal, R_e as before and

N_e: $\Phi_e(C)\neq D$.

We assume $D=\lim_s D_s$ with D_s finite is given by the limit lemma. If F is a finite well generated theory, define $G(F)=\{i\in\omega \mid D_i \text{ or } \overline{D_i}\in F\}$, the *content set.* In this construction we must be very careful that the infinite injury aspect (that is, maximalizing e-states, yet keeping $\Phi_e(C)\neq D$) does not interfere too seriously with the R_e, which boils down to interfering with the $Q_{e,s}$. Roughly speaking, $y\in Q_{e,s}$ must be allowed to contribute $\varepsilon_i B_i^s$ at many stages. For an element of the form $x=\varepsilon_i P_i$ define the *e-state of x at stage $s+1$* to be $b_0\cdots b_e$, a finite sequence of 0's and 1's such that

$$b_i=\begin{cases}1 & \text{if } i\in w_{k,s} \text{ and } k\leqslant e,\\ 0 & \text{otherwise,}\end{cases}$$

for $0\leqslant k\leqslant e$. For completeness we review the definitions given in Soare [20]. Define $a_s=\mu x\,[x\in C_{s+1}-C_s]$ if $C_{s+1}-C_s\neq\emptyset$, and $a_s=\max(C_s\cup\{s\})$ otherwise. We define $\hat{\Phi}_{e,s}(C_s,x)$ to be the same as $\Phi_{e,s}(C_s,x)$ provided that the use function of this computation is $<a_s$ and declare $\hat{\Phi}_{e,s}(C_s,x)$ undefined otherwise. Notice that at a *true stage s*, namely where $C_s\restriction_{a_s}=C\restriction_{a_s}$, any apparent computation is a permanent computation. Let TS = *the set of true stages.* Define $\hat{l}(e,s)=\max\{x:\forall y<x\,(D_s(x)=\hat{\Phi}_{e,s}(C_s,y))\}$ and $\hat{m}(e,s)=\max\{x:u\leqslant s\,(x\leqslant\hat{l}(e,s)$ and $\forall y\leqslant s\,(A_s\restriction_{u(e,y,C_v v)}=A_v\restriction_{u(e,y,C_v v)})\}$. Finally the *restraint* $\hat{r}(e,s)=\max\{\mu(e,x,C_s,s):x\leqslant\hat{m}(e,s)\}$. The injury set is $\hat{I}_e=\bigcup_s\hat{I}_{e,s}$ where $\hat{I}_{e,s}=\{x\mid\exists v\leqslant s\,(x\leqslant\hat{r}(e,v)$ and $x\in C_{s+1}-C_s\}$.

We introduce a recursive function as follows.

$$g(e,s,x)=\begin{cases}1 & \text{if } x\in W_{e,s+1} \text{ and } \varepsilon_i B_{i,s} \text{ occurs in } x\\ & \text{for } G(\varepsilon_i B_{i,s})>\max\{G(B_{e,s}),\hat{r}(y,s)\mid y\leqslant e\},\\ -1 & \text{otherwise.}\end{cases}$$

We say R_e *requires attention at stage* $s+1$ via y if

(i) Either (a) $y\in Q_{e,s}$ and $g(e,s,y)=1$ or (b) $y\in W_{e,s+1}$ and $y\in(T_s,Q_{e,s})^*$,
(ii) $0\notin(T_s,W_{e,s})^*$, and
(iii) e is least.

Let E be a finite subset of W_e. We say x is *W_e-least for E* if $x\in E$ and

$$\forall q\in(E-\{x\})\;\forall t\in\omega\;\;(q\in W_{e,t}\rightarrow x\in W_{e,t})$$

(we may assume $\operatorname{card}(W_{e,s+1}-W_{e,s})\leqslant 1$). We have requirements $\{M_e\}$ which assert that C is maximal. We say M_e *requires attention* at stage $s+1$ if $\exists k\in\bigcup_{f\leqslant s}w_{f,s+1}$ such that $k\geqslant e$ and there exists $q\in\bigcup_{f\leqslant s}w_{f,s+1}$ such that $q>k$ and

(i) $B_{q,s}$ is in a higher e-state than $B_{k,s}$ at stage $s+1$,
(ii) $G(B_{k,s})>\max\{e,\hat{r}(x,s)\mid x\geqslant s\}$.

Construction

Stage $s+1$. For simplicity we adopt the following 'computer science' convention: "$X \leftarrow W \cup Y$" means we are renaming the set $X \cup Y$ by X. In the construction the set T_s is renamed a (finite) number of times. Each time it is renamed, it generates a new collection of $\{B_{i,s}\}$ (where, recall, the $B_{i,s}$ list in order $\{P_i \mid P_i, \bar{P}_i \notin T_s\}$). This could be avoided by extra subscripts and a more complex definition for M_e to require attention. We also adopt a 'subroutine' convention (substage 2) for a 'condition controlled loop'.

Substage 1. If no R_e requires attention for $e<s$ set $T'_s = T_s$ and go to substage 2. If R_e requires attention let y be W_e lest for $\{x \in W_{e,s+1} \mid R_e$ requires attention via $x\}$. Define

$$L(e, s, y) = \{\varepsilon_i\overline{B_{i,s}} \mid \varepsilon_i B_{i,s} \text{ occurs in } y,\ i>e, \text{ and } G(\varepsilon_i B_{i,s}) > \max\{e, \hat{r}(x, s) \mid x \geqslant e\}\}.$$

Now define $T'_s = (T_s, L(e, s, y))^*$ and set

$$Q_{e,s+1} = \begin{cases} Q_{e,s} & \text{if } y \in Q_{e,s}, \\ Q_{e,s} \cup \{y\} & \text{if } y \notin Q_{e,s}, \end{cases}$$

and go to substage 2.

Substage 2 (Begin subroutine).

Case 1*: Subcase* (a). If $T_s \subseteq T'_s$ and no M_f for $f<s$ requires attention, set $T_{s+1} = T'_s$ and go to stage $s+2$.

Subcase (b). If $T'_s \nsubseteq T_s$ and no M_f for $f<s$ requires attention, set $T_{s+1} = T_s$ and go to stage $s+2$.

Case 2*: Subcase* (a). If $T_s \subseteq T'_s$ and $f<s$ is least such that M_f requires attention, find the least (q, k) for f and define

$$T_s \leftarrow (T_s, \{B_{j,s} \mid k \leqslant j < q \ \&\ B_j \notin \{B_{i,s} \mid B_{i,s} \text{ or } \overline{B_{i,s}} \in T'_s\}\})^*.$$

Generate a new set $\{B_{i,s} \mid i \leqslant s\}$ with this T_s, and go to *begin substage 2.*

Subcase (b). If $T_s \nsupseteq T'_s$ and $f<s$ is least such that M_f requires attention, first find (q, k) least for f and put

$$T_s \leftarrow (T_s, B_{j,s} \mid k \leqslant j < q)^*.$$

Generate a new set $\{B_{i,s} \mid i \leqslant s\}$ and go to *begin substage 2.*

Set $T = \bigcup_s T_s$ and $Q_e = \bigcup_s Q_{e,s}$. □ End of Construction

Basically, we maximize all the e-states (for $e \leqslant s$) we possibly can at stage $s+1$, because there are only finitely many e-states, and the ordering of the e-states ensures that case 2 of substage 2 can be applied at most finitely often, after which we apply case 1 and go to stage $s+2$.

We now assume the *injury lemma*: (Soare [20]) If $G \nleqslant_T \hat{I}_e$, then $D \neq \Phi_e(C)$ and the *window lemma*: (Soare [20]) If $D \neq \Phi_e(C)$, then for all $i \leqslant e$, $\lim_{t \in TS} \hat{R}(e, t) < \infty$ where $\hat{R}(i, s) = \max\{r(i, s) \mid i \leqslant e\}$ and so for all $i \leqslant e$, $\liminf_t \hat{R}(e, t) < \infty$. We finally need

Lemma 4.6. *For all e, $\hat{I}_e$ is recursive, R_e requires attention at most finitely often, all the M_e and N_e are met, and all the R_e are met.*

Proof. By simultaneous induction, suppose $\hat{I}_e$ recursive. Then $D \neq \Phi_e(C)$ by the injury lemma, and so $\liminf \hat{R}(e, s) = \hat{R}(e)$ exists by the window lemma. Now at some true stage t, $\forall s > t\, (\hat{R}(e, s) \geqslant \hat{R}(e) = \hat{R}(e, t))$ and $\forall t' \in \mathrm{TS}\, (t' > t$ implies $\hat{R}(i, t') = R(i, t)$ for all $i \leqslant e)$. At some true stage $t' > t$ we thus know $\forall s > t'\, (B_{z,s} = B_z)$ for all $z \leqslant \hat{R}(e)$.

The $\{R_f : g \leqslant e\}$ can require attention at most finitely often thereafter, precisely as before since each time we must pick a new boolean combination of a subset of $B_0, \ldots, B_{\hat{R}(e)}$, and since whenever $x \in Q_{e,s}$ at some stage $s > t'$, a simple induction (by the definition of W_e-least, and requiring attention) ensures that at some *true* stage $t'' \geqslant s$, R_e requires attention *via x*, and is so trimmed, as usual, to a boolean combination of $B_0, \ldots, B_{\hat{R}(e)}$ and so requires attention finitely often. The reader is asked to supply the details. Once the $\{R_f : f \leqslant e\}$ stop requiring attention at stage t'', $\hat{I}_{e+1,s}$ for $s > t$ can change only through the action of M_f for some $f \leqslant e$. We are free to meet the M_f for $f \leqslant e$, and consequently the usual argument of Soare [20, Proposition 3.8], shows that all the M_f for $f \leqslant e$ are met; and the set of elements contributed to C by the M_f for $f \leqslant e$ after stage t'' is simply maximizing f-states, and so $\hat{I}_{e+1}$ is recursive. Thus $\forall f \leqslant e\ (\hat{I}_f$ recursive$\rightarrow R_e$ requires attention at most finitely often, all the R_e, M_e and N_e are met, and $\hat{I}_{e+1}$ is recursive) and so Lemma 4.6 follows. □

Clearly one can blend in other requirements. We conjecture, however, that

(i) If $T = (P_i, \overline{P}_j \mid i \in A, j \in B)^*$ is Martin–Pour-El, then $A \cup B$ is contained in a maximal r.e. set.

(ii) If $\boldsymbol{\delta}$ is a high r.e. degree, then $\boldsymbol{\delta}$ contains an r.e. Martin–Pour-El theory (that is, does the domination of the computation function allow us to use Martin permitting in some way?). (See Note added in proof.)

One might attempt to refute (ii) by showing that the degrees containing Martin–Pour-El theories (or indeed, theories with few r.e. extensions, see Section 5) coincide with some well known class of r.e. degrees. After the high/low degrees, the ones which naturally spring to mind are the promptly simple degrees, that is, those which are not halves of minimal pairs. We give a quick sketch of a proof that:

Theorem 4.7. *There exist minimal pairs of (high) r.e. Martin–Pour-El theories.*

Proof (sketch). In a minimal pair construction, we would satisfy the usual R_e and requirements of the form $\Phi_e(A \cup B) = \Psi_e(A' \cup B') = f$ and f total implies f recursive. When arranged properly (on a tree, say) these requirements co-operate in a way that they

(i) eventually settle down and impose finitely much restraint to the whole construction, or

(ii) impose essentially no restraint.

With the idea that $y \in Q_e$ again may contribute $\varepsilon_i B_i^s$'s at various stages of the construction we simply guess that the current restraint is permanent, and then if we reach a recovery, we allow Q_e to again contribute so that it 'corrects' itself to our current guess. The details are a straight-forward generalization of the minimal pair argument, with modifications along the above lines to allow for the 'lim inf' rather than 'lim'. □

Such considerations do suggest a further question: Are the r.e. degrees containing Martin–Pour-El theories closed upwards? (indeed a filter?).

Evidence for this is the following:

Remark 4.8. *Suppose* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *are r.e. degrees containing Martin–Pour-El theories. Then so does* $\boldsymbol{a} \vee \boldsymbol{b}$*, the least upper bound of* $\boldsymbol{a}$ *and* $\boldsymbol{b}$*.*

Proof. Let $T_k = \langle P_i, \overline{P}_j \mid i \in A_k,\ j \in B_k \rangle$ for $k = 1,\ 2$ be r.e. Martin–Pour-El theories of degrees $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively. Let B_1 and B_2 be r.e. boolean algebras recursively isomorphic to Q/T_1 and Q/T_2 respectively. Let $B_3 = B_1 \oplus B_2$ and let T_3 be the r.e. theory with Q/T_3 recursively isomorphic to B_3. Then T_3 has the desired properties. □

5. Theories with few r.e. extensions

In this section we wish to analyse theories with the property that they have few r.e. extensions, and see their relationship with Martin–Pour-El theories. Recall that T has *few r.e. extensions* if T is essentially undecidable and every r.e. extension of T is principal over T. It is obviously easy to produce r.e. theories with this property which are not Martin–Pour-El, namely let $T = (P_i, \overline{P}_j \mid i \in A, j \in B)^*$ be Martin–Pour-El, let $m,\ n \in \omega - (A \cup B)$ and then $T' = (T, P_m \vee P_n)^*$ has the desired property. Of course T' is contained in a Martin–Pour-El theory (which is principal over T'). This is not always the case.

Theorem 5.1. *There is an r.e. theory T with few r.e. extensions contained in no r.e. well generated theory, and so, in particular, no Martin–Pour-El theory.*

Proof. We build $T = \bigcup_s T_s$ in stages. Let $\{F_e\}$ be a listing of the r.e. well generated theories, that is $F_e = (P_i, \overline{P}_j \mid i \in A_e, j \in B_e)^*$ where $\{A_e, B_e\}$ is a listing of the r.e. disjoint pairs of sets. To ensure that T is contained in no r.e. well generated theory, we meet

N_e: $T \nsubseteq F_e$.

We must also meet the following

R_e: $0 \notin (W_e, T)^* \to \exists x\, ((W_e, T)^* = (T, x)^*)$.

Define $x \oplus y = (x \wedge y) \vee (\bar{x} \wedge \bar{y})$. Notice that $(x \oplus y, x)^* = (x, y)^*$ and $(x \oplus y, \bar{x})^* = (\bar{x}, \bar{y})^*$. The idea involved in satisfying N_e above is to place $x_e \oplus y_e$ into T at some stage, and then wait until x_e or $\bar{x}_e$ occurs in R_e. If neither occurs, then $F_e \nsupseteq T$. If, say x_e occurs, we add $\overline{y_e}$ to T, forcing $\overline{x_e}$ into T, thereby ensuring $(F_e, T)^* \vdash 0$. The interaction of the N_e ensures that we must choose $x_e = P_i$, $y_e = P_j$, say and $\{x_e, y_e\} \cap \{x_i, y_i \mid i \neq e\} = \emptyset$. The interaction of the R_e with the N_e means that we must have a sequence of $x_e \oplus y_e$ say $x_{e,s} \oplus y_{e,s}$ such that $\lim_s (x_{e,s} \oplus y_{e,s})$ exists. Therefore we meet

N_e: $\lim_x x_{e,s} = x_e$ and $\lim_s y_{e,s} = y_e$ exist, with the properties outlined above.

A further source of trouble is that if we attempt to meet the R_e in a similar way as before we want to put $\varepsilon_i P_i$ into $T_{s+1} - T_s$, for $\varepsilon_i P_i$ occurring in some $x \in (T_s, Q_{j,s})^*$ say. The problem is that $\varepsilon_i P_i$ may equal $x_{e,s}$, $\bar{x}_{e,s}$, $y_{e,s}$ or $\bar{y}_{e,s}$ for some e of lower priority. This is, of course, feasible (with finite injury) if $\varepsilon_i P_i = x_{e,s}$ (or $\bar{x}_{e,s}$) provided that for no j does $\varepsilon_j P_j = \bar{y}_{e,s}$ (respectively $y_{e,s}$). But we claim that once $x_{e,s} \oplus y_{e,s}$ occurs in T_s, this case cannot occur. This follows since

$$\begin{aligned}(T_s, x_{e,s} \oplus y_{e,s})^* &= (T_s, (x_{e,s} \vee \overline{y_{e,s}}) \wedge (\overline{x_{e,s}} \vee y_{e,s}))^* \\ &= (T_s, (x_{e,s} \vee \overline{y_{e,s}}), (\overline{x_{e,s}} \vee y_{e,s}))^*\end{aligned}$$

and *so if* $x = \bigvee \varepsilon_i P_i$, *then* $x \notin (T_s, Q_{e,s})$ *means that* $\varepsilon x_{e,s}$ *and* $\overline{\varepsilon y_{e,s}}$ *cannot both occur in* x (for, as above $x \in T_s$, a contradiction). This observation allows us to satisfy the R_j and not seriously injure any N_j in the process.

We introduce some notation:

$B_{i,s}$ will list in order $\{P_i \mid P_i, \overline{P_i} \notin T_s\}$,

as before. We use a certain restraint function $r(e, s)$. We say R_e *requires attention* if there exists $x \in W_{e,s}$ such that $x \notin (T_s, Q_{e,s})$ and $0 \notin (T_s, W_{e,s})^*$. We say N_e requires attention if $x_{e,s}$ is undefined, or $\varepsilon x_{e,s}$ or $\varepsilon y_{e,s}$ occurs in $F_{e,s}$, and N_e is not met.

Priority ranking: $N_0, R_0, N_1, R_1, \ldots$.

Construction

Stage 0. Define $r(e, 0) = e$ for all $e \in \omega$, set $T_0 = \emptyset$ and declare all the $x_{e,s}$, $y_{e,s}$ as undefined.

Stage s + 1. Do one of the following for the requirement of highest priority.

Case 1. If R_e for $e \leqslant s$ requires attention via x define

$$L(e, s, x) = \{\overline{\varepsilon_i B_{i,s}} \mid \varepsilon_i B_{i,s} \text{ occurs in } x \text{ for } i > r(e, s)\}.$$

Set $T_{s+1} = (T_s, L(e, s, x))^*$. Define $r(e, s+1) = r(i, s)$ for $i \leq e$. Declare $r(k, s+1) = r(e, s) + k$ for $k > e$ and declare all the $x_{e,s}$, $y_{e,s}$ for $i \geq e$ as undefined.

Case 2. If N_e is of highest priority $e \geq s$ to require attention and $x_{e,s}$ is undefined, find the first pair $(B_{i,s}, B_{i+1,s})$ such that $i > e$, and $B_{i,s}$ is so large that it has not occurred in any T_s, $Q_{j,s}$ or $W_{j,s}$ and is bigger than $r(e-1, s)$ so far. Now set $x_{e,s} = B_{i,s}$; $y_{e,s} = B_{i+1,s}$ and set $T_{s+1} = (T_s, x_{e,s} \oplus y_{e,s})^*$. Now set $r(e, s+1) = r(e, s) + i + 1$ and for all $k > e$, $r(k, s+1) = r(e, s+1) + k$, and for all $j > e$, set $r(j, s) + 1 = r(j, s)$.

Case 3. If N_e is of highest priority and requires attention and $x_{e,s}$ is defined, then if $\varepsilon x_{e,s} \in F_{e,s}$, we set $T_{s+1} = (T_s, \overline{\varepsilon x_{e,s}})^*$ in this case. If $\varepsilon y_{e,s} \in F_{e,s}$ and $\varepsilon x_{e,s} \notin F_{e,s}$, set $T_{s+1} = (T_s, \overline{\varepsilon y_{e,s}})^*$. Now declare F_e as *met* and note N_e will never again require attention. Keep all the restraints at their current levels.

To complete the construction set $T = \bigcup_s T_s$. □ End of Construction

We need to show that (i) $\lim_s r(e, s) = r(e)$ exists, (ii) each R_e requires attention at most finitely often and (iii) all the R_e are met all the N_e are met.

Certainly it is true that if x is acted on at stage $s+1$ by R_e, then there exists y, a boolean combination of $B_{0,s}, \ldots, B_{r(e,s),s}$ such that

$$T_{s+1} \vdash x \leftrightarrow y,$$

for the same reasons as before. By induction find a stage t where for all $s > t$, for all $i < e$, $j \leq e$

(i) R_i never again requires attention (i.e. is met),

(ii) N_j is met,

(iii) $r(i, s) = r(i, t) = r(i)$.

(Notice, by induction $r(i+1) > r(i)$.)

We claim R_e may require attention at most finitely often hereafter. Again this follows by the proof in Theorem 3.1. Once R_e has finished requiring attention, we are free to define $x_{e+1,s}$ and $y_{e+1,s}$ which will thereafter be protected unless one of $\varepsilon x_{e+1,s}$ or $\varepsilon y_{e+1,s}$ occurs in $F_{e+1,s}$ in which case, its complement is added to T_{s+1}. In any case once R_e has finished requiring attention N_{e+1} needs attention at most twice more, and so is met. Thus $\lim_s r(e+1, s) = r(e)$ exists and the result follows by induction. □

Let us make some remarks concerning the above construction. It is easy to see that T may be generated

(i) as $T = (P_i, \overline{P_j}, P_k \oplus P_n \mid i \in A,\ j \in B,\ k \neq n \text{ and } k, n \in C)^*$,

(ii) with A, B, and C pairwise disjoint.

Alternatively we may consider the collection $\{P_k \oplus P_n\}$ above as $Z = \{P_k \vee \overline{P_n}, P_n \vee \overline{P_k}\}$ as in the proof, with, for all $y \in T$ if $y \in Z$ (here $y = \bigvee \varepsilon_i P_i$), $y \not< x$ for any $x \in Z$. In any case $T \geq_T A \cup B \cup C$ and hence we can produce by the previous techniques T of degree $\mathbf{0}'$. For other degree controlling techniques we may work along the same lines as our preceding result except we must be slightly more subtle. We give an example:

Theorem 5.2. *There exists an r.e. theory T with few r.e. extensions such that* deg(T) *is low, and T is not contained in any r.e. Martin–Pour-El theory.*

Proof. The difference between this and the results so far is that we must control the whole of T_s rather than just $A \cup B$. One way to achieve this is to define $T'_s = \{x \in (T_s)^* \mid x \leqslant s\}$ and then refrain from adding elements to T_s which change some restrained region of T'_s. Another idea is to control T by controlling A, B, and C above directly (notice that C is not r.e. here) in the sense that we control $T'_s = \{P_i, \overline{P}_j, P_k \oplus P_n\}_s$ at stage s, (with the appropriate meanings here for this set). In any case, we define a restraint $r(e, s)$ with $r(e, s) = 0$ if $\Phi_{e,s}(T'_s; e)\uparrow$ and $r(e, s) = u(e, T'_s, s, e)$ if $\Phi_{e,s}(T'_s; e)\downarrow$. Now define $R(e, s) = \max\{e, r(i, s) \mid i \leqslant e\}$. The point is, with priority e we must ensure that

$$T'_{s+1}\restriction_{R(e,s)} = T'_s\restriction_{R(e,s)}.$$

A moment's thought reveals that by choosing our $x_{e+1,s}$, $y_{e+1,s}$ appropriately this can be ensured by eventually showing $\lim_s \hat{R}(e, s) = \hat{R}(e)$ exists, where $\hat{R}(e, s) = \max_{i \leqslant e} \hat{r}(i, s)$ where $\hat{r}(i, s) = \{\max(R(i, s), \hat{z}(i, s)\}$, where $\hat{z}(i, s) = \min\{k \mid$ for all ε_i, and for finite sets F with $y \in F \to y > k$, $(T_s, \varepsilon_i B_{i,s})^*_{i \in F}\restriction_{R(e,s)} = (T_s)^*\restriction_{R(e,s)}\}$, and then only adding $\varepsilon_i B_{i,s}$ into $T_{s+1} - T_s$ for $i > \hat{R}(e, s)$. Now $\hat{z}(i, s)$ may be effectively computed and the rest of the result goes through as before. □

With considerable increase in technical complication along the above lines we may also produce such a T of incomplete high r.e. degree, etc. We now devote our attention to analysing the results 4.3 and 4.4 in this more general setting. Recall that an r.e. theory T has *relatively few r.e. extensions* if every r.e. theory containing T has a common principal extension with T and T is essentially undecidable. It is really fairly obvious that we may use a similar technique to that of 4.3, (avoiding elements of the form $x_{e,s}$ and $y_{e,s}$), to produce an r.e. theory T with relatively few r.e. extensions which does not have few r.e. extensions, and is contained in no r.e. Martin–Pour-El theory.

We have some more serious technical problems when we try to apply our work to Theorem 4.4:

Theorem 5.3. *Below any given r.e. nonzero degree, there is a nonzero r.e. degree which bounds no r.e. theory with few r.e. extensions.*

Proof (sketch). Again, our strategy will be to produce a linked system $z(\langle e, k\rangle)$ of elements which are independent over $W_e \cup Y(e)$ (with notation as in Theorem 4.4). (As with 4.4 this obviously blends with permission). That is, we ensure that for all k,

$$z\langle e, k\rangle \notin (W_e, Y(e), z(\langle e, t\rangle) \mid t \neq k)^*.$$

The problems involved are mildly complicated by the fact that W_e may be no longer generated by, say $A_e \cup B_e$. Let $m_s = \max\{x \in W_{e,s}\}$ and we may consider $W_{e,s}$ as given by $W_{e,s} = \{x \mid x \in (W_{e,s})^* \text{ and } x \leqslant \max\{s, m_s\}\}$. Now, our new negative requirements are:

N_e: If $\Phi_e(D) = W_e$, then if W_e is essentially undecidable, then for all k, $\lim_s z(\langle e, k\rangle, s) = z(\langle e, k\rangle)$ exists, and the system $\{z\langle e, i\rangle \mid i \in \omega\}$ is independent over $(W_e, Y(e))^*$.

Our strategy is this, we wait until a length of agreement is achieved so large that there exists a set $x(\langle e, k\rangle, 0, s), \ldots, x(\langle e, k\rangle, g(e, k), s)$ which is included in this length, is currently independent over $(Y(e, s), W_{e,s})^*$, and is 'good' for n, namely:

(i) If $l(e, s)$ is the current length of $\Phi_{e,s}(D_s) = W_{e,s}$, then $l(e, s) > \max\{$all boolean combinations of $v(\langle e, k\rangle, s)$ and $x(\langle e, k\rangle, i, s) \mid i \leqslant g(e, k)\}$, and

(ii) For all $i \geqslant 1$, $v(\langle e, k\rangle, s) \vee \bigvee_{j \leqslant i} x(\langle e, k\rangle, j, s) \notin (Y(e, s), W_{e,s}, z(\langle e, 0\rangle, s), \ldots, z(\langle e, k-1\rangle, s), v(\langle e, k\rangle, s) \vee \bigvee_{j \leqslant i+1} x(\langle e, k\rangle, j, s))^*$.

We then proceed as before, namely, we define

$$z(\langle e, k\rangle, s) = v(\langle e, k\rangle, s) \vee \bigvee_i x(\langle e, k\rangle, i, s),$$

$$v(\langle e, k+1\rangle, s) = v(\langle e, k\rangle, s \vee \bigvee_{i \neq g(e,k)} x(\langle e, k\rangle, i, s)$$

and

$$n(\langle e, k+1\rangle, s) = v(\langle e, k\rangle, s) \vee \bigvee_{i \langle g(e,k)-1} x(\langle e, k\rangle, i, s).$$

Now restrain on the use of the computation as before.

If we are successful in our restraint, we will ensure that the $z(\langle e, 0\rangle, s), \ldots, z(\langle e, k\rangle, s)$ will be fixed, and remain independent over $(W_{e,s}, Y(e, s))^*$. The reader may check that the remaining details go through almost precisely as in 4.4. Notice that if E_e is essentially undecidable, then such $x(e)$'s as above must occur at some stage. □

We close this section with a theorem and some questions. The theorem analyses how other r.e. theories relate to 'maximal' ones. A natural question to ask is "Does every r.e. theory have an extension T which is 'maximal' in the sense that it is either complete and decidable, or T has few r.e. extensions?" The answer is negative:

Theorem 5.4. *Suppose T is an r.e. weakly Martin–Pour-El theory that is not Martin–Pour-El. Then T is an essentially undecidable r.e. theory contained in no* *r.e. theory with few r.e. extensions. Consequently, if* $\boldsymbol{\delta}$ *is any nonzero r.e. degree, there exists an r.e. essentially undecidable theory of degree* $\boldsymbol{\delta}$ *such that no extension is an r.e. theory with few r.e. extensions.*

Proof. We show that

Lemma 5.5. *If T is an r.e. Martin–Pour-El theory with $T=(P_i, \overline{P_j} \mid i \in A, j \in B)^*$ and $k \in A$, then $T_1=(P_i, \overline{P_j} \mid i \in A-\{k\}, j \in B)^*$ is also an r.e. Martin–Pour-El theory.*

Once we show that 5.5 holds, Theorem 5.4 is deduced as follows: Let T_2 be an r.e. weakly Martin–Pour-El theory of degree $\boldsymbol{\delta}$ that is not Martin–Pour-El. Let T_3 be an r.e. extension with few r.e. extensions. As T_2 is weakly Martin–Pour-El, T_2 and T_3 have a common consistent principal extension, say $0 \notin (T_2, y)^* = (T_3, y)^*$. Now $y = \bigwedge_{i \in F} y_i$ where $y_i = \bigvee_{j \in I_i} \varepsilon_{j(j)} P_{j(i)}$. That is $(T_2, \vec{y})^* = (T_2, y_i \mid i \in F)^*$. Now we may delete all occurrences of $\varepsilon_j P_j$'s which occur in T_2 since

(i) if $\overline{\varepsilon_{j(i)} P_{j(i)}} \in T_2$, then $\overline{y_i \in T_2}$;

(ii) if $\overline{\varepsilon_{j(i)} P_{j(i)}} \in T_2$, then $\overline{\varepsilon_{j(i)} P_{j(i)}} \wedge y_i \in T_2$ and so

$$\text{if } y_i' = \bigvee_{\substack{k \in I_j \\ k \neq j(i)}} \vee\, \varepsilon_{k(i)} P_{k(i)}, \quad \text{then } T_2 \vdash y_i \leftrightarrow y_i'.$$

This gives us a 'minimal' set of generators $y_1, \ldots, y_n$, say, where no occurrence of $\varepsilon_{j(i)} P_{j(i)}$ in any y_i is an element of T_2. Now it is easy to see that we may generate *finite* sets A', B' such that $y_1, \ldots, y_n \in (P_i, \overline{P_j} \mid i \in A', j \in B')^* = T_4$ such that if $T_2 = (P_i, \overline{P_j} \mid i \in A_2, j \in B_2)^*$, then $A' \cap A_2 = B' \cap B_2 = \emptyset$, and $A' \cap B_2 = B' \cap A_2 = \emptyset$. Therefore T_2, $T_3 \subseteq (T_2, y)^* \subseteq (P_i, \overline{P_j} \mid i \in A', j \in B')^* = T_4$. Now as T_3 has few r.e. extensions, T_4 has few r.e. extensions and so T_4 is Martin Pour-el, and this contradicts Lemma 5.5 since $\operatorname{card}((A_2 \cup B_2 \cup A' \cup B') - (A \cup B)) < \infty$. $\square$

Proof of Lemma 5.5. Let T, T_1 be as described. Again consider only elements in $\bigvee \varepsilon_i P_i$ form. Let $W \supseteq T_1$ and W be nonprincipal over T_1. Let

$$W_1 = ((P_k)^* \cap W) \cup T_1)^*,$$
$$W_2 = (((\overline{P_k})^* \cap W) \cup T_1)^*,$$
$$W_3 = ((W - (W_1 \cup W_2))^* \cup T_1)^*.$$

It is easy to see W_1, W_2, and W_3 are all r.e., and that $W = (W_1 \cup W_2 \cup W_3)^*$. We claim W_1 is principal over T_1. If $P_k \in W_1$ stop, since $W_1 = (T_1, P_k)^*$. If $P_k \notin W_1$ consider W_1' which results from the deletion of all occurrences of P_k from elements in W_1, and then generating the theory (again, r.e.). Now consider $W_1'' = (W_1', T)^*$. Certainly $0 \notin W_1''$ and so $W_1'' = (T, y_1, \ldots, y_n)^*$ where $y_i = \bigvee \varepsilon_j P_j$ where $\varepsilon_j P_j$, $\overline{\varepsilon_j P_j} \notin T$. Thus it is easy to see that under these circumstances $(T_1, P_k \vee y_1, \ldots, P_k \vee y_n)^* = W_1$. Similarly one may show W_2 is principal over T_1.

Finally one can show along similar lines that $W_3 = (T, z_1, \ldots, z_n)$ where z_j is of the form $z_j = \bigvee \varepsilon_i P_i$ and $\varepsilon_i P_i \notin T$, $\overline{\varepsilon_i P_i} \notin T$, and $z_j \in W$ for all j. Then $W_e = (T_1, z_1, \ldots, z_n)^*$ and so W_3 is also principal over T_1. Thus $W_i = (T_1, f_i)^*$ say, and so $W = (T_1, \bigwedge f_i)^*$ and so W is principal over T_1, a contradiction. □

This generates:

(i) The obvious question — Is it the case that every r.e. essentially undecidable theory has an r.e. extension with *relatively* few r.e. extensions? What about well-generated theories? (We suspect that if we try to sharpen this to say "Is it the case that every r.e. essentially undecidable well generated theory is contained in an r.e. weakly Martin–Pour-El theory" it is surely false, but we can probably produce with an e-state construction an r.e. pair of r.e. sets contained in no r.e. maximal pair.)

(ii) Also our results indicate the similarities of these r.e. theories with maximal r.e. sets in a certain sense, so it is natural to ask about automorphisms of these objects in the lattice of r.e. theories. In dealing with algebraic structures one must be very careful about such matters (cf. [4], [6], or [18]). For example: Are all r.e. theories with few r.e. extensions automorphic? Notice that the failure of $=^*$ to be an equivalence relation means that there is no meaningful way to address "the lattice modulo $=^*$" as in the r.e. set case.

6. Theories with decidable extensions

So far we have concerned ourselves only with essentially undecidable r.e. theories. Of course, the other situation deserves some attention and, in this section, we shall give some ideas towards an analysis of this situation. To begin with, the results and techniques of Jockusch–Soare [11, 12] are obviously relevant here. For example, let T be r.e. Martin–Pour-El, and consider this as an r.e. filter in the free boolean algebra Q. Let B be a boolean algebra recursively isomorphic to $Q \bmod T$. Let $B' = B \times \{0, 1\}$, the boolean algebra formed by the product of B with the 2-element boolean algebra. Now find an r.e. theory T' with $Q \bmod T'$ recursively isomorphic to B'. Then of course every r.e. extension of T' is principal over T', T' has $2^{\aleph_0}$ complete extensions and precisely one decidable complete extension (generalizing Theorem II of Martin and Pour-El [14]). The reader is referred to [11, 12], for further results along these lines.

There is another natural way to view r.e. theories with decidable extensions by considering the lattice $L(D)$ or r.e. subtheories of a complete decidable theory D. Since they are all recursively (auto-) isomorphic, we may take D as $(P_i \mid i \in \omega)^*$. Thus we shall ask questions about the way r.e. subtheories of D relate to one another. One apparently very natural way to study $L(D)$ is via $=^*$ for *amongst* members of $L(D)$ this is an equivalence relation:

Convention. In this section $T =^* W$ will mean there exists $x \in D$ such that $(T, x)^* = (W, x)^*$.

We might hope that $L(D)$ under $=^*$ might act like various other lattices of r.e. substructures which have already been studied. For example, suppose we call $M \in L(D)$ a *maximal* r.e. theory if $M \neq^* D$ and for all $W \in L(D)$ if $W \supset M$ then either $W =^* M$ or $W =^* D$. We have:

Theorem 6.1. *Let K be a maximal r.e. set. Then $M = (P_i \mid i \in K)^*$ is a maximal r.e. theory.*

Proof. Suppose $W \supset M$, $W \neq^* M$, $W \neq^* D$ and $W \in L(D)$. We construct $D(W) = \bigcup_s D_s(W)$. (The technique is due to Downey [3, 6] in other settings.)

Construction
Stage 0. Set $b_{i,s} = P_i$ for all $i \in \omega$ and $D_0(W) = \emptyset$.
Stage s + 1. Let e be the first element, if any, $\leqslant s$ such that
(i) $\forall i < e\ (\forall j \leqslant i\ (b_{j,s} \notin (W_s, b_{k,s} \mid k < j)^*))$, and
(ii) $b_{e,s} \in (W_s, b_{j,s} \mid j < e)^*$.
If no such e exists, then set $D_{s+1}(W) = D_s(W)$. If e exists set $D_{s+1}(W) = D_s(W) \cup \{b_{e,s}\}$ and in this case set $b_{i,s+1} = b_{i,s}$ for $i < e$ and $b_{i,s+1} = b_{i+1,s}$ for $i \geqslant e$. □ End of Construction

We leave the reader to show (cf. [3]), setting $i(D(W)) = \{i \mid P_i \in D(W)\}$, that
(a) $\mathrm{card}(\omega - i(D(W))) = \infty$ (lest $W =^* D$),
(b) $i(D(W)) \supset K$ (as $M \subset W$),
(c) $\mathrm{card}(i(D(W)) - K) = \infty$ (lest $W =^* M$),
and this contradicts the maximality of K. □

Various other results and techniques borrowed from, say, the lattice of r.e. sets seem to apply (see here Remmel [17, 18]). In so far as the lattice of r.e. sets is concerned, we can be somewhat more shrewd. Let $Q \in L(D)$; then we define the *interval lattice* $L(Q, D)$ to be lattice of r.e. theories T with $Q \subset T \subset D$. We may show:

Theorem 6.2. *There exists a decidable theory Q such that $L(Q, D)$ is recursively isomorphic to $L(\omega)$ the lattice of r.e. sets.*

Proof. Let N denote a recursive copy of the boolean algebra of finite and cofinite sets with $A(N)$ the set of atoms recursive. Let $\{q_0, q_1, \ldots\}$ be a recursive listing of $A(N)$ without repetitions. Now for $C \in L(\omega)$ define $I(C)$ denote the ideal generated by $\{q_i \mid i \in C\}$. For any r.e. ideal J of N with $J \subset I(\omega)$ = the ideal generated by $A(N)$, J is generated by an r.e. subset of the set of atoms. Thus if we define a map $i \to q_i$, this induces a recursive isomorphism between $L(\omega)$ and $H = \{I \mid I \text{ is an r.e. ideal of } N \text{ and } I \subset I(\omega)\}$.

Now let Q be a recursive ideal of $\boldsymbol{Q}$ with $\boldsymbol{Q}$ mod Q recursively isomorphic to N. Let $\Omega(\boldsymbol{Q}/Q) \cong N$ denote this recursive isomorphism. Now let P be the recursive prime ideal of Q with $\Omega(P) \cong I(\omega)$. Our observations ensure that the lattice of r.e. ideals containing Q and contained in P are recursively isomorphic to $L(\omega)$. We dualize to filters, since the lattice of r.e. filters of $\boldsymbol{Q}$ is recursively isomorphic to the lattice of r.e. ideals of $\boldsymbol{Q}$ (by reversing $\wedge$ to $\vee$), and the result follows. □

Since $L(Q, D)$ is elementarily definable in the lattice of r.e. theories, it follows that the first-order theory of the lattice of r.e. theories is undecidable, as the first-order theory of the lattice of r.e. has been shown undecidable by Hermann [9, 7, 8] (also by Harrington [unpublished] by representing boolean pairs with parameters).

Another form of analysis of $L(D)$ is to examine features of $L(D)$ which do not have analogues in the lattice of r.e. sets. For example, inspired by results in matroids (cf., e.g. [6]) we might conjecture the existence of a *supermaximal* theory, that is $M \in L(D)$ such that $M \neq^* D$ and for all $W \in L(D)$ if $W \subset D$ then either $W =^* D$ or $W = D$. Alas this is not the case:

Theorem 6.3. *There exist no r.e. supermaximal theories.*

Proof. Define a sequence $K = \{q_0, q_1, \ldots\}$ in stages.

Stage 0. Set $q_0 = P_0$.

Stage s + 1. Set $q_{s+1} = P_{s+1} \vee \bigvee_{j \leq s} \overline{P_j}$.

Note that for all $j \in \omega$, $(q_j)^* \cap (K - \{q_j\}^* = \{1\}$, and $(K)^* = D$. Suppose M is supermaximal. As $M \neq^* D$, there exists $i \in \omega$ such that $q_i \notin M$. By supermaximality, if $Q = (K - \{q_i\})^*$, then either $(Q, M)^* = D$ or $(Q, M)^* =^* M$. Now if $(Q, M)^* =^* M$, as $Q =^* D$, it follows that $M =^* D$. Therefore $(Q, M)^* = D$. Therefore for some $m \in M$, $x \in Q$, and $y \in D$

$$\begin{aligned} q_i &= (m \wedge x) \vee y \\ &= m \wedge \Big(\bigwedge_{j \neq 1} q_j \vee z\Big) \vee y, \quad \text{for some } x \in D \text{ since } x \in Q \\ &= \Big(m \wedge \bigwedge_{j \neq i} q_j\Big) \vee y', \quad \text{for some } y' \in D. \end{aligned}$$

Therefore $q_i \vee \bar{m} = \bigwedge_{j \neq i} q_j \vee \bar{m} \vee y' \in Q$ and so $q_i \vee \bar{m} = 1$ as $(q_i)^* \cap Q = \{1\}$. The point is that this means $q_i \in M$ since $m \wedge q_i = m \wedge (q_i \vee \bar{m}) = m \in M$, a contradiction. □

Significant differences between $L(\omega)$ and $L(D)$ *do* occur however. We finish with an example of this (actually inspired by the vector space case [16]). An r.e. set of independent axioms for an r.e. theory T is an r.e. set of generators $\{g_i \mid i \in I\} = G$ such that $T = (\{g_i \mid i \in I\})^*$, and yet $T \neq (G')^*$ for any proper subset G' of G. Formally we say the set $\{g_i \mid i \in I\}$ is an (*independent*) *axiomatization* if for all i, $g_i \notin (\{g_j \mid j \in I - \{i\}\})^*$. We show that the usual processes of extending axioms fails in a strong way. To sharpen our results, we say an r.e. set of axioms $\{g_i \mid i \in I\}$ is a *strong axiomatization* if, for all i, $(g_i)^* \cap (\{g_j \mid j \neq i\})^*$. It is easy to show that every axiomatization may be refined to a strong axiomatization, namely if $\{g_i \mid i \in I\}$ is a set of axioms, then $\{g_i' \mid i \in I\}$ is a set of strong axioms where

(i) $g_0' = g_0$, and $g_{i+1}' = g_{i+1} \vee \bigvee_{j \leq i} \overline{g_i'}$,

(ii) for all n, $(g_0, \ldots, g_n)^* = (g_0', \ldots, g_n')^*$.

It is fairly straightforward to show that every r.e. theory T has a recursive strong axiomatization.

When it comes to extending such sets we say an r.e. set A of strong axioms is *nonextendible in D* if $A \subset D$ and

(a) $(A)^* \neq^* D$.

(b) For all r.e. strong axiomatizations B, if $B \subset D$, then $A \subset B$ implies $(A)^* =^* (B)^*$.

In particular, A cannot be extended to an r.e. axiomatization for D. We have

Theorem 6.4. *There exists an r.e. theory $T \in L(D)$ such that*

(i) *T has an r.e. strong axiomatization contained in a strong axiomatization for D.*

(ii) *Every r.e. strong axiomatization of T is nonextendible in D.*

Proof. Let I_e denote the e-th r.e. strong axiomatization. We build a strong axiomatization $J = \bigcup_s J_s$ in stages so that $T = (J)^*$ has the desired properties. At each stage s we specify an r.e. set $\{b_{j,s} \mid j \in \omega\}$ with the idea that

$$N_e\colon\quad \lim_s b_{e,s} = b_e \text{ exists}$$

and $J \cup \{b_e \mid e \in \omega\}$ is a strong axiomatization for D. The positive requirements are

$$R_e\colon\quad (I_e)^* \supset J \text{ and } T \neq^* (I_e)^* \text{ implies } (I_e \cap T)^* \neq T.$$

To meet the R_e we shall define elements $x = x(e, s)$ and $y = y(e, s)$ with the idea of putting a nontrivial element $z = z(e, s) = (x \oplus y) \vee k$ into T and keep x and y out of T in such a way that $x \in (T, y)^*$ which will ensure that if $(I_e \cap T)^* = T$, then I_e cannot be a strong axiomatization. We say R_e requires attention if $e \leq s$ is least such that $x(e, s)$ is currently undefined and

(i) $(I_{e,s+1} \cap T_s)$ is a strong set of axioms.

(ii) There exist x, $y \in I_{e,s}$ such that

(a) $x \notin (T_s, b_{i,s}, y, x(j,s), y(j,s) \mid i \leq e, j < e \text{ and } x(j,s) \text{ defined})^*$,
(b) $y \notin (T_s, b_{i,s}, x, x(j,s), y(j,s) \mid i \leq e, j \leq e, x(j,s) \text{ defined})^*$.

Construction
Stage 0. Set $b_{i,0} = q_i$ where q_i as given in Theorem 6.3. Declare all the $x(i, 0)$, $y(i, 0)$ undefined.
Stage s + 1. If no R_e requires attention do nothing. If R_e requires attention via x, y define $x(e, s+1) = x$, $y(e, s+1) = y$, declare all $x(i, s+1)$, $y(i, s+1)$ for $i > e$ undefined, maintain the $x(j, s+1)$, $y(j, s+1)$ for $i < e$ and define

$$z(e, s+1) = x \oplus y \vee \bigvee_{i \leq e} \overline{b_{i,s}} \vee \bigvee_{z \in J_s} \bar{z} \vee \bigvee_{x(i,s) \text{ defined}, i<e} \overline{x(i,s)}$$
$$\vee \bigvee_{y(i,s) \text{ defined}, i<e} \overline{y(i,s)}.$$

Set $J_{s+1} = J_s \cup \{z(e, s+1)\}$ and define $T_{s+1} = (J_{s+1})^*$. Finally set

$$b_{i,s+1} = \begin{cases} b_{i,s} & \text{for } i \leq e, \\ P_{k(i)} \vee \bigvee_{j<i} b_{j,s+1} \vee \bigvee_{z \in J_{s+1}} \bar{z} & \text{for } i > e, \end{cases}$$

where $P_{k(i)}$ is the least P_t with $P_t \notin (J_{s+1}, b_{j,s+1} \mid j < i)^*$. □ End of Construction

Lemma 6.5. *Each R_e requires attention at most finitely often, all the N_e are met and $J \cup \{b_i \mid i \in \omega\}$ is a strong axiomatization for D.*

Proof. By induction, suppose $J_s \cup \{b_{i,s} \mid i \in \omega\}$ is a strong axiomatization for D, and $J_{s+1} = J_s \cup \{z(e, s+1)\}$, with

$$z_{s+1} = z(e, s+1) = x \oplus y \vee \bigvee_{i \leq e} \overline{b_{i,s}} \vee \bigvee_{z \in J_s} \bar{z} \vee \bigvee_{i<e} \overline{x(i,s)} \vee \bigvee_{i<e} \overline{y(i,s)}.$$

Clearly $x \oplus y \notin (J_s, b_{0,s}, \ldots, b_{e,s})^*$ lest $x \in (T_s, b_{0,s}, \ldots, b_{e,s}, y)^*$. If $q \in (J_s, b_{0,s}, \ldots, b_{e,s})^* \cap (z_{s+1})^*$, then

$$q = z_{s+1} \vee q_1 = \Big(\bigwedge_{z \in J_s} z \wedge \bigwedge_{i \leq e} b_{i,s}\Big) \vee q_2,$$

say. Therefore,

$$q = \Big(\bigwedge_{z \in J_s} z \wedge \bigwedge_{i \leq e} b_{i,s}\Big) \vee z_{s+1} \vee q_1 \vee q_2 = 1.$$

Now suppose $q \in (J_{s+1}, b_{i,s} \mid i \leq e, i \neq j)^* \cap (b_{j,s+1})^*$. Then

$$q = b_{j,s} \vee q_1 = \Big(\bigwedge_{z \in J_s} z \wedge z_{s+1} \wedge \bigwedge_{i \neq j} b_{i,s}\Big) \vee q_2.$$

Thus

$$q = b_{j,s} \vee q_1 \vee b_{j,s} = [\cdots] \vee b_{j,s}$$
$$= \Big(\bigwedge_{z\in J_s} (z \vee b_{j,s}) \wedge (z_{s+1} \vee b_{j,s}) \wedge \bigwedge_{i\neq j} (b_{i,s} \vee b_{j,s})\Big) \vee q_2$$
$$= \Big(\bigwedge_{z\in J_s} z \wedge \bigwedge_{i\neq j} b_{i,s}\Big) \vee q_2 \vee b_{j,s}.$$

Therefore $q \in (b_{j,s})^* \cap (T_s, b_{i,s} \mid i \leq e, i \neq j)^* = \{1\}$. Therefore $J_{s+1}, b_{0,s}, \ldots, b_{e,s}$ is a strong axiomatization. The $b_{i,s}$ for $i > e$ are similar, and the result follows. □

Lemma 6.6. *All the R_e are met.*

Proof. We first show with priority e that our action

(i) does not 'injure' any R_j for $j < e$,

(ii) temporarily satisfies R_e.

For (i) suppose R_e is attacked via (x, y, z_{s+1}) say. Suppose for $i < e$, $x(i, s) \in T_{s+1} - T_s$. Then

$$x(i,s) = \Big(\bigwedge_{z\in J_s} z \wedge \Big((x \oplus y) \vee \bigvee_{i\leq e} \overline{b_{i,s}} \vee \bigvee_{z\in J_s} \bar{z} \vee \bigvee_{i<e} \overline{x(i,s)} \vee \bigvee_{i<e} \overline{y(i,s)}\Big)\Big) \vee q$$

so that

$$x(i,s) = x(i,s) \vee x(i,s) = [\cdots] \vee x(i,s) = \Big(\bigwedge_{z\in J_s} z\Big) \vee x(i,s) \vee q \in T_s,$$

a contradiction.

For (ii) suppose we have $\lim_s x(e,s) = x(e)$ exists, say at stage t, and $I_e \supset T$ and $(I_e \cap T)^* = T$. Notice $(x)^* \cap (T \cup \{y\})^* \neq \{1\}$. That is, let $q_1, \ldots, q_n \in I_e \cap T$ with $z_t \in (q_1, \ldots, q_n)^*$. Then $(q_1, \ldots, q_n, x, y)^*$ should be a strong set of axioms. However define

$$q = y \wedge z_t = \Big(((x \vee \bar{y}) \wedge (y \vee \bar{x})) \vee \Big[\bigvee_{i\leq e} \overline{b_i} \vee \bigvee_{z\in J_t} \bar{z} \vee l\Big]\Big) \wedge y, \quad \text{say}$$
$$= (x \wedge y) \vee ([\cdots] \wedge y)$$
$$= (x \vee ([\cdots] \wedge y)) \wedge (y \vee ([\cdots] \wedge y)).$$

Now if we put $q' = (x \vee ([\cdots] \wedge y))$, then $q' = 1$, since $q' \in (q_1, \ldots, q_n, y)^* \cap (x)^*$. Moreover notice by the above, since $(y \vee ([\cdots] \wedge y)) = y$, that $q = y \wedge z_t = q' \wedge y = y$, that is $y = y \wedge z_t$. This means $z_t \in (y)^*$. Consequently, since $z_t \in T$, it follows that $z_t = 1$ (as $z_t \in (q_1, \ldots, q_n)^* \cap (y)^*$). This means $(x \vee \bar{y} \vee [\cdots]) \wedge (y \vee \bar{x} \vee [\cdots]) = 1$, so that $x \vee \bar{y} \vee [\cdots] = 1$ and $y \vee \bar{x} \vee [\cdots] = 1$. We claim that this implies

$$x \in (J_t, b_0, \ldots, b_e, x(i,t), y(i,t), y \mid i < e \text{ and } x(i,t) = x(i) \text{ defined})^* = M.$$

This follows since $q'' \in M$ where

$$q'' = (x \vee \bar{y} \vee [\cdots]) \wedge \bigwedge_{z \in J_t} z \wedge \bigwedge_{i>e} x(i) \wedge \bigwedge_{i<e} y(i) \wedge \bigwedge_{j \leq e} b_j \wedge y$$

$$= x \wedge \bigwedge_{z \in J_t} z \wedge \bigwedge_{i<e} x(i) \wedge \bigwedge_{i<e} y(i) \wedge \bigwedge_{j \leq e} b_j \wedge y$$

and this implies $x \in M$, contradicting the definition of 'requiring attention', and so (ii) is established.

Finally we must show that if $(I_e)^* \supset T$ and $(I_e^*) \neq^* T$, then R_e will require attention. If R_e fails to require attention at some large stage t (by which all the R_j for $j < e$ have settled down), we know we have a finite fixed set

$$F = \{x(0), \ldots, x(e-1), y(0), \ldots, y(e-1) \mid x(i), y(i) \text{ defined (at } t)\}.$$

Now as $(I_e)^* \neq T$, there exist infinitely many x and y in I_e with

$$x \notin (T, b_0, \ldots, b_e, F, y)^* \quad \text{and} \quad y \notin (T, b_0, \ldots, b_e, F)^*.$$

For each such y, x, it follows that

$$y \in (T, b_0, \ldots, b_e, F, x)^*,$$

lest R_e requires attention. This means that for each such x, y,

$$y = \Big(t \wedge \bigwedge_{i \leq e} b_i \wedge f \wedge x\Big) \vee g, \quad \text{say} \qquad \Big(f = \bigwedge_{p \in F} p\Big).$$

Therefore $y \vee \bigvee_{i \leq e} \bar{b}_i \vee \bar{f} \in (T, x)^*$. Now choosing x, y, as above ensures that $\{x, y\} \cap T = \emptyset$. If $(I_e \cap T)^* = T$, and I_e is a strong axiomatization, it follows that $y \vee \bigvee_{i \leq e} \bar{b}_i \vee \bar{f} = 1$, since $y \vee \bigvee_{i \leq e} \bar{b}_i \vee \bar{f} \in (I_e - \{y\})^* \cap (y)^* = 1$. Therefore,

$$\Big(\bigwedge_{i \leq e} b_i \wedge f\Big) = \Big(\bigvee_{i \leq e} \bar{b}_i \vee \bar{f} \vee y\Big) \wedge \Big(\bigwedge_{i \leq e} b_i \wedge f\Big) = y \wedge \bigwedge_{i \leq e} b_i \wedge f,$$

and hence, $y \in (T, b_0, \ldots, b_e, F)^*$ a contradiction. □

At this stage we close with a few remarks. Evidently the last argument may be extended in various ways. For example, we could control the degree of T (by permitting and coding, via the $\{b_{i,s} \mid i \in \omega\}$), or we could place various lattice-theoretic restrictions on T by weaving lattice type requirements in. One idea is to analyse hypersimplicity, maximality etc. to see if they blend with nonextendibility. We leave these constructions to the interested reader. It seems fairly clear, however, that the lattice of r.e. theories and $L(D)$ both have an unusually rich structure, for which the problems do not reduce to any of the lattices of r.e. substructures previously studied. We close with a problem: Describe a nontrivial class of automorphic invariants in either of the above lattices, and how many automorphisms do they have?

7. Some concluding remarks

We may wish to analyse the results of the preceding sections and search for connections with other structures and lattices. Some areas here are ideals in commutative rings, comes of orderings of formally real fields etc. Two we wish to mention are recursively bounded Π_1^0 classes and the lattice of r.e. sets.

Identifying r.e. theories with r.e. ideals in the free boolean algebra Q, as we know there is an effective inclusion inverting 1–1 correspondence between r.e. ideals and recursively bounded Π_1^0 classes (under Zariski topology) in the Stone space 2^ω. In view of this, we know that there is a natural way of assigning degrees of Π_1^0 classes. What sort of Π_1^0 class corresponds to an r.e. theory with few r.e. extensions? Answer: a *thin* Π_1^0 class.

Definition. We say a Π_1^0 class (in 2^ω) C is *thin* if

(i) C is infinite and for each Π_1^0 subclass C' of C, either $C' = \emptyset$ or $C' = C \cap Q$ for some clopen subset Q of 2^ω.

(ii) C contains no recursive members.

Similarly

Definition. We say a Π_1^0 class (in 2^ω) C is *semi-thin* if

(i) It contains no recursive members.

(ii) If C' is a Π_1^0 class with $C' \subset C$, then there exists a nonempty Π_1^0 class D and a clopen subset Q with $D = C \cap Q = C' \cap Q$.

If I is an ideal of Q let $\Pi(I)$ denote the Π_1^0 class associated with I and similarly $I(\Pi)$ the ideal associated with Π of course $I(\Pi(I)) = I$. Much of the analysis of I has been by analysis of $\Pi(I)$. We hope our results indicate that one may obtain interesting insights by analysis of I directly. These results also indicate that we might be able to obtain sharper results in effective algebra. For example the cone of orderings of a recursive group is a Π_1^0 class. These results suggest that there exist recursive infinitely generated orderable (abelian) groups whose only recursively orderable subgroups are finitely generated. This is indeed the case (although the proof techniques differ significantly (see Downey–Kurtz, "Recursion theory and Ordered Groups", Ann. Pure Appl. Logic 32 (1986) 137–151)).

The techniques described here also appear to indicate new features of the lattice of r.e. sets. Let us define an r.e. set A to be *g-hypersimple,* if $\text{card}(\omega - A) = \infty$ and for all r.e. sequences $\{D_x\}_{x \in W}$ of canonical finite sets there exist finite sets $X = \{x_1, \ldots, x_n\} \subset W$ and G such that for all $y \in W$, there exists $x \in X$ such that $D_x \subset A \cup G$ and $D_y \cap G = D_x \cap G$. Similarly we could define *g-hh-simplicity* by replacing 'canonical finite sets' by 'finite sets'. Clearly g-h-simplicity implies h-simplicity. Is every hh-simple set g-h-simple? Do g-hh-simple sets exist? Here, our results can clearly be modified to establish the existence of g-h-simple sets in high or low degrees, establish h-simple sets which are not g-h-simple in each r.e. degree, an r.e. degree which bounds no r.e. g-h-simple set etc.

It is unclear how the above notions relate to the currently analysed ones. G-hh-simple sets remain to be explored. In particular we would like to know: if every hh-simple r.e. set is g-h-simple, then is g-h-simplicity lattice-definable in the r.e. sets? (Also, what about g-hh-simplicity?) An affirmative solution would be particularly interesting, since then we would have a class of r.e. sets nontrivially splitting the low degrees including all hh-simple sets which is still invariant under automorphisms (and therefore include sets whose lattice of r.e. supersets is not a boolean algebra — a class that has resisted investigation so far).

Note added in proof

Carl Jockusch, Mike Stob and the author have shown that the degrees containing Martin–Pour-El theories are exactly the array non-recursive r.e. degrees. This class is closed upward and corresponds to, roughly speaking, those degrees that arise in arguments which need 'multiple permitting'. These results will appear in a forthcoming paper entitled "Array non-recursive sets and multiple permitting arguments".

References

[0] M. Bickford and C. Mills, Lowness properties of r.e. sets, J. Symbolic Logic, to appear.

[1] R.G. Downey, Abstract dependence, recursion theory and the lattice of recursively enumerable filters, Ph.D. Thesis, Monash University, Victoria, Australia (1982).

[2] R.G. Downey, On the lattice of r.e. filters axiomatizable theories and Π_1^0 classes, Preprint, Monash Logic series 36 (1981).

[3] R.G. Downey, On a question of A. Retzlaff, Z. Math. Logik. Grundl. Math., 29 (1983) 379–384.

[4] R.G. Downey and G.R. Hird, Automorphisms of supermaximal subspaces, J. Symbolic Logic 50 (1985) 1–9.

[5] R.G. Downey, Nowhere simplicity in matroids, J. Austral. Math. Soc. (Ser. A) (1983) 28–45.

[6] R.G. Downey, Co-immune subspaces and complementation in V_∞, J. Symbolic Logic 49 (1984) 528–538.

[7] E. Hermann, Orbits of hyperhypersimple sets and the lattice of Σ_3^0 sets, J. Symbolic Logic, to appear.

[8] E. Hermann, Definable structures in the lattice of recursively enumerable sets, to appear.

[9] E. Hermann, Definable boolean pairs in the lattice of recursively enumerable sets, to appear.

[10] C. Jockusch and R. A. Shore, Pseudo jump operators I: the r.e. case, Trans. A.M.S. 275 (1983) 599–609.

[11] C. Jockusch and R.I. Soare, Π_1^0 classes and degrees of theories, Trans. A.M.S. 173 (1972) 33–56.

[12] C. Jockusch and R.I. Soare, Degrees of members of Π_1^0 classes, Pacific J. Math. 40 (1972) 605–616.

[13] D.A. Martin, Classes of recursively enumerable sets and degrees of unsolvability, Z. Math. Logik. Grundl. Math. 12 (1966) 295–310.

[14] D.A. Martin and M.B. Pour-El, Axiomatizable theories with few axiomatizable extensions, J. Symbolic Logic 35 (1970) 205–209.
[15] G. Metakides and A. Nerode, Recursion theory and algebra, in: J.N. Crossley, ed., Algebra and Logic, Lecture Notes in Math. 450 (Springer, Berlin, 1975) 209–219.
[16] G. Metakides and A. Nerode, Recursively enumerable vector spaces, Ann. Math. Logic 11 (1977) 147–171.
[17] J.B. Remmel, recursively enumerable boolean algebras, Ann. Math. Logic 14 (1978) 75–107.
[18] J.B. Remmel, recursion theory on algebraic structures with an independent set, Ann. Math. Logic 18 (1980) 153–191.
[19] H. Rogers, Theory of Recursive Functions and Effective Computability (McGraw-Hill, New York, 1967).
[20] R.I. Soare, The infinite injury priority method, J. Symbolic Logic 41 (1976) 513–530.
[21] R. Smullyan, Theory of Formal Systems, Ann. Math. Studies 47 (Princeton University Press, Princeton, NJ, 1961).